J. Myanmar Acad. Arts Sci. 2020 **Vol. XVIII.No.2**B

NONCOMMUTIVITY AND CUTOFF SCALES

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Abstract

Basic ideas and requirements of noncommutative space are presented. Position cutoff and momentum cutoff are calculated by observing the commutation relations. Relation between elements of noncommutative algebra and Dirac operator is also discussed.

Keywords: noncommutative space, position cutoff, momentum cutoff

Introduction

The presence of Plank constant in quantum mechanics and speed of light imply a short wavelength photon will have a high momentum. The proper way to the formalism of quantum mechanics is to define noncommuting operators acting on Hilbert space and a physical state given by a state vector. Restrictions will arise even if momentum is ignored and insert gravitational constant in the formalism of quantum mechanics. There are reasons to believe that problems in unifying quantum mechanics and gravity can be solved by developing the aspects of noncommutative geometry.

A noncommutative algebra is an algebraic structure in which the principle binary operation is not commutative. Additional structures are allowed to carry by the noncommutative algebra of functions.

Noncommutative geometry is a subject of mathematics that concerned with a geometrical approach to noncommutative algebra. It is a system consisting of a Hilbert space, a noncommutative algebra of operators acting on Hilbert space and a self adjoint operator. Noncommutative geometry has been inspired by quantum mechanics and has applications to number theory, deformation theory, quantum field theory, elementary particle physics and solid state physics.

Noncommutivity

A manifold is a locally Euclidean continuous space. It is a space with coordinates that locally looks like Euclidean but globally can bend and distort. Differential manifold is continuous and differentiable. It is a set that can be continuously parameterized. Various objects can exist on a manifold. Infinitely differentiable functions are included in these objects. If M is a manifold, the group of all infinitely differentiable functions is called $C^{\infty}(M)$.These functions can be added and multiplied.

$$(f+g)(x) = f(x) + g(x) \qquad (1)$$

$$(fg)(x) = f(x)g(x) \qquad (2)$$

Hence, $C^{\infty}(M)$ is a commutative algebra. Geometrical objects, such as, vectors and tensor fields can be defined in terms of the algebra $C^{\infty}(M)$.

For n dimensional manifold, a vector **V** with

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$$\mathbf{V} \rightarrow (V^1(x), \ldots, V^n(x)) \tag{3}$$

where

$$x = x^1, \ldots, x^n \tag{4}$$

will follow local coordinates transformation rules.

If an operator V is defined as

$$\mathrm{V} = V^1 \frac{\partial}{\partial x^1} + \cdots + V^n \frac{\partial}{\partial x^n} \tag{5}$$

then it is clear that V is a linear operator from $C^\infty(M)$ into itself. And it satisfies the Leibniz relation

$$\mathrm{V}(fg) = \mathrm{V}(f)g + f\mathrm{V}(\mathrm{g}) \tag{6}$$

for every f and g.

So, vector fields, differential forms and general tensor fields are derivations of $C^\infty(M)$.

Theorem I Two manifolds M and N are diffeomorphic if and only if the algebra of functions $C^\infty(M)$ and $C^\infty(N)$ are isomorphic.

So, the commutative algebra of the infinitely differentiable functions on M, $C^\infty(M) := A$, encodes all the differential geometric properties of M. And this leads to basic ideas of noncommutative geometry.

The dimension of a space can be derived from the rate of growth of the eigenvalues of D^2. Dimension can be defined if (7) does not diverge for single value of d.

$$\lim_{x \to \infty} \frac{N_x}{x^{0.5d}} \tag{7}$$

Regularity is also required for characteristics of manifolds. That is, a and $[D, a]$ belong to δ^k with

$$\delta(T) = [|D|, T] \tag{8}$$

There should be a J for manifolds, such that

$$[a, Jb^* J^{-1}] = 0 \qquad \forall a, b$$

$$[[D, a], b^0 = Jb^* J^{-1}] = 0 \qquad \forall a, b \tag{9}$$

Noncommutative geometrical objects such as noncommutative vectors, noncommutative differential forms can be defined through a noncommutative algebra. But these noncommutative geometrical objects are only sensible in purely algebrical structure.

Basic ideas lead to noncommutative spaces is based on a theorem which was proved by Gelfand and Naimark.

Theorem II The algebra of continuous functions $C(X)$ of a compact topological space X is a commutative C^* algebra with the complex conjugate $\overline{f}$ is its hermitian adjoint f^*.

Theorem III Given a complex commutative C^* algebra A, one can construct a unique compact topological space X, such that A can be identified with the function algebra $C(X)$.

Topological space is a set of points, along with a set of neighborhood for each point satisfying a set of axioms relating points and neighborhood. Compact topological space means that the topological space is closed and bounded and continuous complex valued functions on a topological space form a commutative algebra.

C^* algebra is an involutive Banach algebra with

$$\left\|a^* a\right\| = \left\|a\right\|^2 \tag{10}$$

for each element a.

Banach algebra is a complete normed algebra and in an involutive algebra, each elements a has properties

$$\left\|a+b\right\| \le \left\|a\right\| + \left\|b\right\| \tag{11}$$

$$\left\|\alpha a\right\| = \left|\alpha\right| \left\|a\right\| \tag{12}$$

$$\left\|ab\right\| \le \left\|a\right\| \left\|b\right\| \tag{13}$$

$$\left\|a^*\right\| = \left\|a\right\| \tag{14}$$

$$\left\|a\right\| = 0 \Leftrightarrow a = 0 \tag{15}$$

Algebra of compact operators on Hilbert space and $n \times n$ matrices are example of C^* algebras. With norm defined as

$$\left\|A\right\|^2 = \max_{eigenvals} A^\dagger A \tag{16}$$

According to theorem of Gelfand and Naimark, there is a one to one correspondence between compact topological spaces and complex commutative C^* algebras. This leads to idea that observing and studying noncommutative C^* algebras amounts to studying noncommutative compact topological spaces.

Momentum and Position Cutoff

To quantize the Minkowski space-time, it will be required that

$$\left[q^\mu, q^\nu\right] = ikq^{\mu\nu} \tag{17}$$

In the limit

$$k \to 0$$

$$q^\nu \to x^\nu$$

And it will be required that q^ν to be Hermitian operators on Hilbert space.

If there is a magnetic field, then the commutation becomes as

$$[p_1, p_2] = i\hbar eB \tag{18}$$

Points of momentum space is replaced by area $\hbar eB$. This allows the cutoff as

$$p^2 \geq \hbar eB \tag{19}$$

Letting $B = 1T$

leads he cutoff scale to $1.7\times10^{-53}\left(kgms^{-1}\right)^2$. Let

$$1.7\times10^{-53}\left(kgms^{-1}\right)^2 = 1\Gamma \tag{20}$$

And it is postulated that the momentum space exhibits the noncommutative structure under 1Γ. For the position commutation

$$[x_1, x_2] = i\tau \tag{21}$$

then, τ in (21) must have the dimension of area. Which leads to

$$\tau = \hbar Gc^{-3}$$

$$=2.7\times 10^{-70}m^2 = 1\widehat{M} \tag{22}$$

Again it is postulated that the position space will have the noncommutivity under $1\widehat{M}$.

Operator and Space

The relativistic equation

$$E^2 - p^2 = m^2$$

for a particle with mass m can be quantized into

$$\left(-\frac{\partial^2}{\partial t^2} + \text{V}\right)\Psi = m^2\Psi \tag{23}$$

In order to get a first order, the operator term in (23) must be taken into square root. It will lead to

$$\left(i\varphi - m\right)\Psi = \left(i\gamma^{\mu}\partial_{\mu} - m\right)\Psi = 0 \tag{24}$$

And it is required that

$$\left\{\gamma^{\nu}, \gamma^{\mu}\right\} = 2g^{\nu\mu}$$

to recover (23). And γ^{ν} belongs to $M_4(£)$. Where a matrix representation is

$$\gamma^0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}$$

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad , i = 1, 2, 3 \tag{25}$$

Where

$$\sigma_0 = I$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (26)$$

But the Dirac operators that appear in noncommutative geometry should be in the form

$$D\Psi = \sum_{i=1}^{n} e_i . \nabla_{e_i} \Psi \quad (27)$$

with a local orthonormal frame. Then one can defined Hilbert space and the operator will be a symmetric operator on this space. For any

$$f \in A$$

$$\Psi \in H$$

$$[D, f]\Psi = i^{-1}\left(\frac{d}{d\phi}(f\Psi) - f\frac{d}{d\phi}\Psi\right)$$

$$= i^{-1}\frac{df}{d\phi}\Psi \quad (28)$$

Suppose

$$D = \begin{pmatrix} 0 & m \\ m^* & 0 \end{pmatrix} = |m|F$$

$$F = \begin{bmatrix} 0 & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{bmatrix} \quad (29)$$

Then the commutator is

$$[D, a] = (a_2 - a_1)|m|\begin{bmatrix} 0 & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{bmatrix} \quad (30)$$

Thus, it can be seen that (30) is a discrete derivative.

Conclusion

It is postulated that position space will exhibit noncommutative structure under and momentum space will exhibit noncommutative structure under 1Γ. Classical concepts of points and geometry should be replaced by much more general entities.

Acknowledgements

I wish to express my sincere thanks to Professor Dr Khin Khin Win, Head of Department of Physics, University of Yangon, for her kind approval to conduct this work.

My sincere thanks also go to all of the professors, Department of Physics, University of Yangon, for their kind permission to carry out this work.

Special thanks are due to Dr Thant Zin Naing, Pro-rector (retired), International Theravada Buddhist Missionary University for his helpful suggestion and encouraging.

Last but not least, I would like to thank my father, U Mya Than, Former Demonstrator, Department of Physics, University of Yangon, who also helped me with suggestions and ideas for this work.

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