DYNAMICAL SYSTEMS APPROACH TO FRW MODELS

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Abstract

Dynamical systems approach to FRW models have been presented. Some techniques of non-linear dynamics have been utilized to find evolution equations. Visualization of stable and unstab**le** points have been performed and a brief discussion for the results obtained has been given.

Keywords : Dynamical systems approach, FRW models, non-linear dynamics.

Introduction

All the dark energy models obey the general form of the equation of state$p = \omega(x(z))\rho$ in which the coefficient ω is parameterized by the scale factor x or red shift z. It is interesting to some general properties of an evolutional path of such models. For this aim dynamical system methods seem to be a natural method because it offers the possibility of investigating whole space of solutions starting from all admissible initial conditions. The dynamics of the FRW model with dark energy can be represented in the form of two dimensional dynamical systems, $\dot{x} = P(x, y), \dot{y} = Q(x, y)$where $P, Q \in C^{\infty}$.The phase portraits of the system are organized by critical pointsP $(x_o, y_o) = Q(x_o y_o) = 0$.

Theoretical Background

Dynamical System in General

Any physical or abstract entity whose configuration at any given time can be specified by some set of numbers, system variables, and whose configuration at a later time is uniquely determined by its present and past configurations through a set of rules are often encountered. In continuous-time systems the rules are expressed as equations that specified the time derivatives of the system variables in terms of their current (and possible past) values. In such cases, the system variables are real numbers that varies continuously in time. The continuous-time dynamical system takes the form,

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$$x(t) = f(x(t); p, t)$$

The components of vector xare the system variables and the vector f representsa function of all the system variables at fixed values of the parameters. In discrete-time systems the rules are expressed as equations giving new values of the system variables as function of the current (and possible past) values. In spatially extended systems, each system variable is a continuous function of spatial position as well as time and the equations of motion take the form of partial differential equations

$$\frac{\partial x}{\partial t} = f(x, \nabla x, \nabla x^2, \ldots.; p, t)$$

A set of equations describing a discrete-time dynamical system takes the form,

$$x(t + 1) = F(x(t); p, t)$$

Here the function F directly gives the new x at the next time step, rather than the derivative from which a new x can be calculated. The function F is often referred to as a map that takes the system from one time step to the next. In all cases the evolution of the system is described as a motion instate space of all possible values of the vectorx. A trajectory is a directedpath through state space that shows the values of the system variables at successive times.

Linear and Nonlinear Systems

A linear system is one for which any two solutions of the equations of motion can be combined through simple addition to generate a third solution, given appropriate definition of the zeros of the variables. The way to recognize a linear dynamical system is that its equations of motion will involve only polynomial functions degree one in the system variables; there will be no products of different system variables or nontrivial functions of any individual variable. For all types of linear systems, the constraint that the sum of any two solutions also must be a solution has profound consequences. In a linear stable system, all solutions asymptotically approach the fixed point as time progresses. In an unstable system, all solutions that do not starts exactly on the fixed point diverge from it exponentially at long times. The marginal case, in which the variables neither decay to zero nor diverge, occurs primarily in Hamiltonian systems, in which conservation of energy prohibits convergence or divergence of nearby phase space trajectories. In the nonlinear system, the equations of motion include at least one term that contains the square or higher power, a product of system variables (for more complicated functions or them), or some sort of threshold function, so that the addition of two solutions does not yield a valid new solution, no matter how the system variables are defined. All physical systems describable in terms of classical equations of motion are nonlinear. The quantum mechanical theory of atoms and molecules is a linear theory.

Phase Plane Analysis of Dynamical Systems

For understanding the dynamics of linear and nonlinear systems, the description of its behavior in phase space is quite useful. The two independent variables (x, p_x) here $x, y = \dot{x},$ define the space in which the solution moves. For a particle having only one independent variable, the phase space is only two-dimensional and hence it is often referred to as the phase plane. At any time the value of the phase space co-ordinates (x, y) completely defines the state of the system. A given solution to the equations of motion will map out a smooth curve in the phase plane as a function of time. This is called a phase curve or phase trajectory and the motion along it is called the phase flow. Because of the unique properties of solutions to differential equations, different phase space trajectories do not cross each other. A picture made up of sets of phase curves is called a phase portrait.

Classification of Fixed Points

The nature of phase curves will depend on the eigenvalues of the stability matrix λ_1 *and* λ_2. However, the form of eigenvectors determines the actual directions of the local phase flow. The different possibilities are discussed in this section.

Case (i): $\lambda_1 < \lambda_2 < 0$ − **a stable node.** As the eigenvalues are negative, the local flow decays in both directions determined by D_1 and D_2 into the fixed point, as illustrated in figure (1(a)).

Case (ii): $\lambda_1\lambda_2 > 0 -$ **an unstable node.** The local flow grows exponentially away from the fixed point in both directions, as shown in figure (1(b)).

Case (iii): $\lambda_1 < 0 < \lambda_2 -$ **hyperbolic point or saddle point**. One direction grows exponentially and the other decays exponentially, as illustrated in figure (1(c)). The incoming and outing directions are often referred to as the stable and unstable manifolds of separatrix, respectively.

Case (iv): $\lambda_1 = -\alpha + i\beta, \lambda_2 = -\alpha - i\beta(\alpha, \beta > 0) -$ **a stable spiral point** .Since the two eigenvalues λ_1 and λ_2 have the negative real part $-\alpha$ the flow spirals in toward the fixed points, as shown in figure (1(d)).

Case (v): $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta -$ **an unstable spiral point**. Because of the positive real parts, the flow spirals away from the fixed point (figure (1(e))) will be stable equilibrium point.

Case (vi): $\lambda_1 = i\omega, \lambda_2 = -i\omega -$ **an elliptic point or simply centre**. As the two eigenvalues are purely imaginary, the phase curves will be closed ellipses, as shown in figure (1(f)).This will be stable equilibrium point.

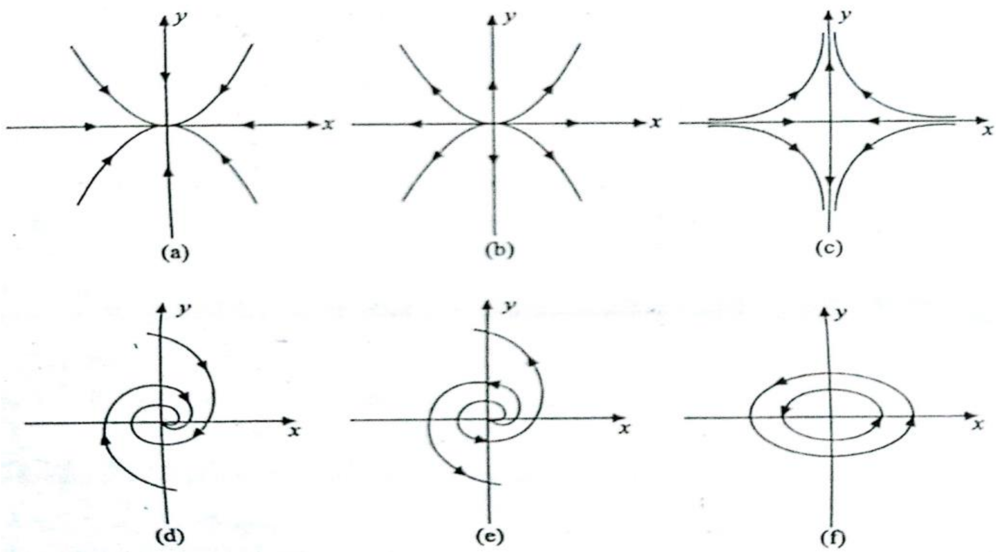

Figure 1: Local phase flows for (a) stable star; (b) unstable star; (c) stable improper node; (d) unstable improper node.

The Friedmann-Robertson-Walker Model with Dynamical Dark Energy

The Friedmann-Robertson-Walker model with source in the form of non –interacting dust matter and dark energy which the equation of state is parameterized by$p = \omega\,(x)\rho$. Then from the conservation condition,

$$\dot{\rho} = -\,3\,H\,(\,\rho + p)$$

The FRW dynamics is

$$H^2 = \frac{\rho}{3} - \frac{k}{x^2}\,, \quad \text{and} \qquad \rho = 3\,\frac{\dot{x}^2}{x^2} + \frac{3}{x^2}k$$

From conservation condition

$$p = -\,\frac{\dot{\rho}}{3}\frac{x}{\dot{x}} - \rho$$

$$\dot{\rho} = 3\,\frac{2\,\dot{x}\ddot{x}}{x^2} + 3\,x^2(-2x^{-3}\dot{x}) + 3\,k\,(-2x^{-3}\dot{x})$$

$$p = -2\,\frac{\ddot{x}}{x} - \frac{\dot{x}^2}{x^2} - \frac{k}{x^2}$$

Then the basic dynamical equations reduce to

$$\dot{x} = y$$

$$\rho + p = 2\,\frac{\rho}{3} - 2\frac{\ddot{x}}{x}$$

Therefore $\quad \ddot{x} = -\frac{1}{6}(\rho + 3p)x$

then one get

$$\dot{y} = -\frac{1}{6}(\rho(x) + 3p(x))x$$

This equation constitutes a two dimensional autonomous dynamical systems.

The General Properties of the FRW Model

The basic dynamical equations reduce to$\dot{x} = y$ and$\dot{y} = -\frac{1}{6}(\rho + 3p)x$, where $p = \omega\rho$.The dynamics of the FRW model with dark energy can be represented in the form of two dimensional systems,

$$\dot{x}=P(x,y),\ \ \dot{y} = Q(x,y)$$

where $P, Q \in C^{\infty}$.

The character (type) of critical points is determined from eigenvalues of the linearization matrix A: $A = \begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{bmatrix}_{(x_0, y_0)}$ where,

$$\frac{\partial P}{\partial x} = \frac{\partial y}{\partial x} = 0 \text{ and } \frac{\partial P}{\partial y} = \frac{\partial y}{\partial y} = 1$$

$$\frac{\partial Q}{\partial x} = \frac{\partial \dot{y}}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{1}{6}\left[\rho(x) + 3\omega\rho(x)\right]x\right) = -\frac{1}{6}\frac{\partial}{\partial x}\left[x\rho(x)[1 + 3\omega(x)]\right]$$

$$\frac{\partial Q}{\partial y} = \frac{\partial \dot{y}}{\partial y} = \frac{\partial y}{\partial y}\left[-\frac{1}{6}\left[\rho(x)[1 + 3\omega(x)]x\right]\right] = 0$$

Therefore, one has

$$A = \left[-\frac{1}{6}\frac{\partial}{\partial x}\left(x\rho(x)(1+3\omega(x)\right)\begin{matrix}1\\0\end{matrix}\right]_{(x_0,y_0)} ,$$

where $x = a$ is dimensionless scale factor expressed in terms of its present value a_0 .In the eigen problem $\det[A-\lambda I]=0$ then reads $\lambda^2 - \lambda TrA + \det A = 0$. Consequently the sign of determent A determines the type of the critical points that is whether λ is real or complex. It is consequence of the fact that,

$$TrA = 0 \text{ and } \det A = \frac{1}{6}\frac{\partial}{\partial x}\left(x\rho(x)\left(1+3\omega(x)\right)\right).$$

Then one gets, the critical points of the system can be saddle points if $\det(A)_{x_0} > 0$ or centers if $\det(A)_{x_0} > 0$ In the first case eigenvalues are real of opposite sign while in the second case they are purely imaginary and conjugated.

Concluding Remarks

The main message of this paper is to note that there exists a systematic methods of classification and investigation the dynamical equation of state in the quite general form, the equation of state$p = \omega\ (x)\rho$. The FRW models with dynamical dark energy modeled in terms of the equation of state$p = \omega\left(x(z)\right)\rho$ in which the coefficientω is parameterized by the scale factor xorz. The advantages of using a complementary description of dynamic as a Hamiltonian flow, the evolution of the model is represented as a motion of unit mass in one –dimensional potential $V(a)$.The properties of potential function V can serve as a toolfor qualitative classification of all evolution paths. The main subject of this paper is presentation how dynamics of the FRW model with dynamical dark energy can be reduced to the form of a two -dimensional dynamical system.

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