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THEORETICAL AND NUMERICAL ASPECTS OF GRAVITATIONAL WAVES

Nyein Thida[1], Hla Hla Maw[2]

Abstract

It has been attempted to give a brief description of gravitational waves and its underlying physics for both theoretical and numerical aspects. First of all, the theoretical background of long sought gravitational waves (GWs) and its related connection to astrophysics and cosmology have been given in detail. Then, the accelerating charge, accelerating mass and amplitude of GWs are theoretically and numerically investigated within the framework of general relativity, classical mechanics and statistical mechanics points of view.

Keywords: gravitational waves, general relativity, dipole and quadrupole moments.

Introduction

Gravitational waves (GWs) are generated by accelerated masses, that propagate as waves outward from their source at the speed of light. They were predicted by Albert Einsteinin 1916on the basis of his general theory of relativity. Gravitational waves transport energy as gravitational radiation, a form of radiant energy similar to electromagnetic radiation. Gravitational wave astronomy is a branch of observational astronomy that uses gravitational waves to collect observational data about sources of detectable gravitational waves such as binary star systems composed of white dwarfs, neutron stars, black holes and events such as supernovae and the formation of the early universe shortly after the Big Bang.

On 11 February 2016, the LIGO and Virgo Scientific Collaboration announced they had made the first observation of gravitational waves. The observation was made five months earlier, on 14 September 2015, using the Advanced LIGO detectors. The gravitational waves originated from a pair of merging black holes. After the initial announcement the LIGO instruments detected two more confirmed, and one potential, gravitational wave events. In August 2017, the two LIGO instruments and the Virgo instrument observed a

1. Dr, Associate Professor, Department of Physics, University of Yangon
2. Lecturer, Department of Physics, University of Yangon.

fourth gravitational wave from merging black holes, and a fifth gravitational wave from abinary neutron starmerger. Several other gravitational wave detectors are planned or under construction.

Gravitational waves can penetrate regions of space that electromagnetic waves cannot. They are able to allow the observation of the merger of black holes and possibly other exotic objects in the distant Universe. Such systems cannot be observed with more traditional means such as optical telescopes or radio telescopes, and so gravitational wave astronomy gives newinsights into the working of the Universe. In principle, gravitational waves could exist at any frequency. However, very low frequency waves would be impossible to detect and there is no credible source for detectable waves of very high frequency. Stephen Hawking and Werner Israel list different frequency bands for gravitational waves that could plausibly be detected, ranging from 10^{-7} Hz up to 10^{11} Hz.

Gravitational waves have two important and unique properties. First, there is no need for any type of matter to be present nearby in order for the waves to be generated by a binary system of uncharged black holes, which would emit no electromagnetic radiation. Second, gravitational waves can pass through any intervening matter without being scattered significantly. Whereas light from distant stars may be blocked out by interstellar dust, for example, gravitational waves will pass through essentially unimpeded. These two features allow gravitational waves to carry information about astronomical phenomena heretofore never observed by humans, and as such represent a revolution in astrophysics.

Effect of Gravitational Waves on Free Particles

To investigate the geodesic equations for the trajectories of material particles and photons in a nearly flat space time, during to the passage of a gravitational wave, one need to study the following.

Proper Distance Between Test Particles

The amplitude of the metric perturbation is described by just two independent constants, A_{xx} and A_{xy}. The physical significance of these constants by examining the effect of the gravitational wave on a free particle, in an initially wave-free region of space time.The free particle's trajectory satisfies the geodesic equation

$$\frac{dU^{\beta}}{d\tau} + \Gamma^{\beta}_{\mu\nu} U^{\mu} U^{\nu} = 0 \qquad (1)$$

where τ is the proper time. The particle is initially at rest, i.e. initially $U^{\beta} = \delta^{\beta}_{t}$.

Thus, the initial acceleration of the particle is

$$\left(\frac{dU^{\beta}}{d\tau}\right)_0 = -\Gamma^{\beta}_{tt} = -\frac{1}{2}\eta^{\alpha\beta}(h_{\alpha t,t} + h_{t\alpha,\alpha} - h_{tt,\alpha}) \qquad (2)$$

However, $A_{\square t} = 0 \Rightarrow h_{\alpha t} = 0$

Also, recall that $h = \bar{h} = 0$. Therefore it follows that $h_{\square t} = 0$ for all α which in turn implies that $\left(\frac{dU^{\beta}}{d\tau}\right)_0 = 0$ (3)

Hence a free particle, initially at rest, will remain at rest indefinitely. However, 'being at rest' in this context simply means that the coordinates of the particle do not change. As the gravitational wave passes, this coordinate system adjusts itself to the ripples in the space time, so that any particles remain 'attached' to their initial coordinate positions. Coordinates are merely frame-dependent labels, however, and do not directly convey any invariant geometrical information about the space time.[5]

The proper distance between the particles is then given by

$$\Delta l = \int \left| g_{\alpha\beta} dx^{\alpha} dx^{\beta} \right|^{1/2} \qquad (4)$$

$$\Delta l = \int_0^{\varepsilon} \left| g_{xx} \right|^{\frac{1}{2}} \cong \sqrt{g_{xx}(x=0)}\ \in \qquad (5)$$

$$g_{xx}(x=0) = \eta_{xx} + h^{(TT)}_{xx}(x=0) \qquad (6)$$

Since $h_{xx}^{(TT)}(x=0)$ in general is not constant, it follows that the proper distance between the particles will change as the gravitational wave passes. It is essentially this change in the proper distance between test particles which gravitational wave detectors attempt to measure.

Geodesic Deviation of Test Particles

The study of the behaviour of test particles are more formally using the idea of geodesic deviation. By defining the vector ξ^α which connects the two particles and for a weak gravitational field, taking proper time approximately equal to coordinate time is

$$\frac{\partial^2 \xi^\alpha}{\partial t^2} = R^\alpha_{\mu\nu\beta} U^\mu U^\nu \xi^\beta \tag{7}$$

where $U^\mu$are the components of the four-velocity of the two particles. Since the particles are initially at rest, then $U^\mu = (1,0,0,0)T$

and $\xi^\beta = (0,\epsilon,0,0)^T$

then simplifies to $$\frac{\partial^2 \xi^\alpha}{\partial t^2} = \in R^\alpha_{ttx} = -\in R^\alpha_{txt} \tag{8}$$

Hence, two particles initially separated by in the x-direction, have a geodesic deviation vector which obeys the differential equations. Similarly, it is straightforward to show that two particles initially separated by in the y-direction, have a geodesic deviation vector which obeys the differential equations.

Ring of Test Particles: Polarisation of Gravitational Waves

To consider the geodesic deviation of two particles, one at the origin and the other initially at coordinates $x = \epsilon \cos\theta$, $y = \epsilon \sin\theta$ and $z = 0$, i.e. in the *x*-*y* plane as a gravitational wave propagates in the *z*-direction. ξ^x and ξ^y obey the differential equations

$$\frac{\partial^2}{\partial t^2}\xi^x = \frac{1}{2}\in \cos\theta \frac{\partial^2}{\partial t^2} h_{xx}^{(TT)} + \frac{1}{2}\in \sin\theta \frac{\partial^2}{\partial t^2} h_{xy}^{(TT)} \tag{9}$$

and $$\frac{\partial^2}{\partial t^2}\xi^y = \frac{1}{2}\in\cos\theta\frac{\partial^2}{\partial t^2}h_{xy}^{(TT)} - \frac{1}{2}\in\sin\theta\frac{\partial^2}{\partial t^2}h_{xx}^{(TT)} \quad (10)$$

We can identify the solution

$$\xi^x = \in\cos\theta + \frac{1}{2}\in\cos\theta A_{xx}^{(TT)}\cos\omega t + \frac{1}{2}\in\sin\theta A_{xy}^{(TT)}\cos\omega t \quad (11)$$

and $$\xi^x = \in\sin\theta + \frac{1}{2}\in\cos\theta A_{xy}^{(TT)}\cos\omega t - \frac{1}{2}\in\sin\theta A_{xx}^{(TT)}\cos\omega t \quad (12)$$

θ varies between *0* and 2π. An initially circular ring of test particles in the x-y plane, initially equidistant from the origin and the metric perturbation has $A_{xx}^{(TT)} \neq 0$ and $A_{xy}^{(TT)} = 0$ .In this case the solution for ξ^x and ξ^y reduces to

$$\xi^x = \in\cos\theta\left(1 + \frac{1}{2}A_{xx}^{(TT)}\cos\omega t\right) \quad (13)$$

$$\xi^y = \in\sin\theta\left(1 - \frac{1}{2}A_{xx}^{(TT)}\cos\omega t\right) \quad (14)$$

In the case of the metric perturbation has $A_{xy}^{(TT)} \neq 0$ and $A_{xx}^{(TT)} = 0$, the solutions for ξ^x and ξ^y reduce to

$$\xi^x = \in\cos\theta + \frac{1}{2}\in\sin\theta\, A_{xy}^{(TT)}\cos\omega t \quad (15)$$

$$\xi^y = \in\sin\theta + \frac{1}{2}\in\cos\theta\, A_{xy}^{(TT)}\cos\omega t \quad (16)$$

To understand the relationship between these solutions and those for $A_{xx}^{(TT)} = 0$,we define new coordinate axes x′and y′ by rotating the x and y axes through an angle of $-\pi/4$, so that

$$x' = \frac{1}{\sqrt{2}}(x - y) \quad (17)$$

$$y' = \frac{1}{\sqrt{2}}(x+y) \tag{18}$$

If we write the solutions for $A_{xy}^{(TT)} \neq 0$ in terms of the new coordinates x' and y' after some algebra we find that

$$\xi'^{x} = \in \cos\left(\theta + \frac{\pi}{4}\right) + \frac{1}{2} \in \sin\theta \left(\theta + \frac{\pi}{4}\right) A_{xy}^{(TT)} \cos \omega t \tag{19}$$

and $$\xi'^{y} = \in \sin\left(\theta + \frac{\pi}{4}\right) + \frac{1}{2} \in \sin\theta \left(\theta + \frac{\pi}{4}\right) A_{xy}^{(TT)} \cos \omega t \tag{20}$$

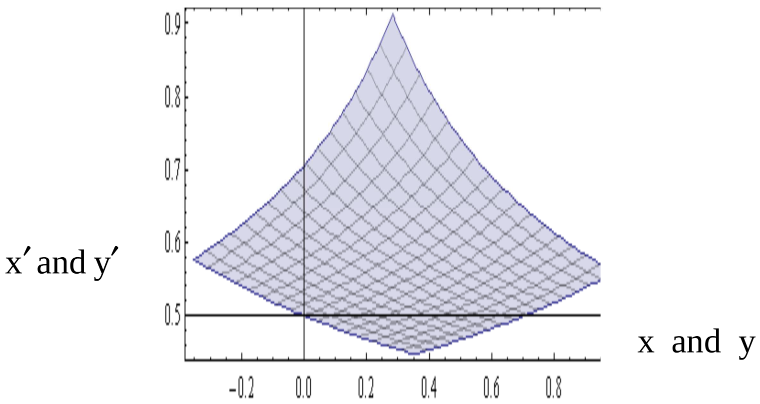

Figure 1: Parametric Plot for the interactive relation between x' and y'

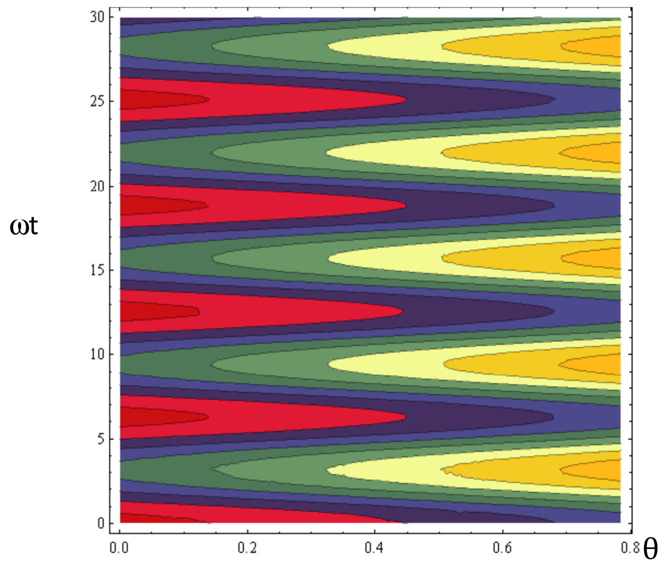

Figure 2: Contour Plot for Gravitational wave of the new coordinate x′ and y′ $A_{xx}^{(TT)} = 1$

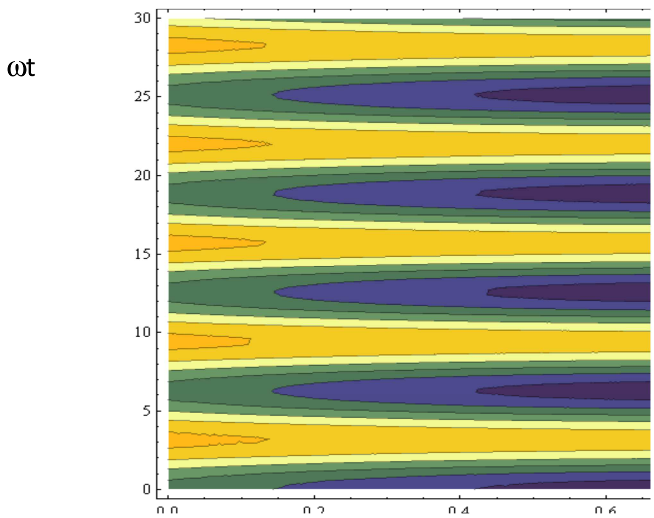

Figure 3: Contour plot for Gravitational waves of the new coordinate x′ and y′ $A_{xy}^{(TT)} = 1$

The Production of Gravitational Waves

The reasons why gravitational radiation is quadrupolar to lowest order, and we estimate the amplitude of the gravitational wave signal from binary neutron star system.

The Quadrupolar Nature of Gravitational Waves

The nature of gravitational radiation by drawing analogies with the formulae that describe electromagnetic radiation. This approach is crude at best since the electromagnetic field is a vector field while the gravitational field is a tensor field, but it is good enough for our present purposes. Essentially, we will take familiar electromagnetic radiation formulae and simply replace the terms which involve the Coulomb force by their gravitational analogues from Newtonian theory.

Electric and Magnetic Dipoles

In electromagnetic theory, the dominant form of radiation from a moving charge or charges is electric dipole radiation. For a single particle (e.g. an electron) of charge *e* with acceleration *a* and dipole moment changing as $\ddot{d} = e\ddot{x} = ea$, the power output or luminosity is given by

$$L_{\text{electric dipole}} \propto e^2 a^2 \qquad (23)$$

For a general distribution of charges, with net dipole moment *d*, the luminosity is

$$L_{\text{electric dipole}} \propto e^2 \ddot{d}^2 \qquad (24)$$

The next strongest types of electromagnetic radiation are magnetic dipole and electric quadrupole radiation. For a general distribution of charges, the luminosity arising from magnetic dipole radiation is proportional to the second time derivative of the magnetic dipole moment, i.e.

$$L_{\text{magnetic dipole}} \propto \ddot{\mu} \qquad (25)$$

where μ is given by a sum (or integral) over a distribution of charges:-

$$\mu = \sum_{q_i} (\text{position of } q_i) \times (\text{current due to } q_i) \qquad (26)$$

Gravitational Analogues

The gravitational analogue of the electric dipole moment is the mass dipole moment *d* summed over a distribution of particles, *{Ai}*

$$d = \sum_{A_i}(m_i x_i) \tag{27}$$

where m_i is the rest mass and x_i is the position of particle A_i.

The luminosity of gravitational 'mass dipole' radiation should be proportional to the second time derivative of *d*. However, the first time derivative of *d* is

$$\dot{d} = \sum_{A_i}(x_i)\dot{x}_i = P \tag{28}$$

where *p* is the total linear momentum of the system. Since the total momentum is conserved, it follows that the gravitational 'mass dipole' luminosity is zero i.e there can be no mass dipole radiation from any source. Similarly, the gravitational analogue of the magnetic dipole moment is

$$\mu = \sum_{A_i}(x_i)\times(m_i v_i) \equiv J \tag{29}$$

where *J* is the total angular momentum of the system. Since the total angular momentum is also conserved, again it follows that the gravitational analogue of magnetic dipole radiation must have zero luminosity. Hence there can be no dipole radiation of any sort from a gravitational source. The simplest form of gravitational radiation which has non-zero luminosity is, therefore, quadrupolar.

Example: ABinary Neutron Star System

Finally, the example of the gravitational wave signature of a particular astrophysical system: a binary neutron star. In general it can be called slow motion approximation for a weak metric perturbation $h_{\mu}\nu<<1$ then for a source at distance *r*

$$h_{\mu\nu} = \frac{2G}{c^4 r}\ddot{I}_{\mu\nu} \tag{30}$$

Where $I_{\mu\nu}$ is the reduced quadrupole moment defined as

$$I_{\mu\nu} = \int \rho(\vec{r})(x_\mu x_\nu - \frac{1}{3}\delta_{\mu\nu} r^2)dV \qquad (31)$$

Consider a binary neutron star system consisting of two stars both of Schwarzschild mass *M*, in a circular orbit of coordinate radius *R* and orbital frequency *f*. For simplicity we define our coordinate system so that the orbital plane of the pulsars lies in the x-y plane, and at coordinate time $t = 0$, the two pulsars lie along the x-axis.

$$I_{xx} = 2MR^2[\cos^2(2\pi ft) - \frac{1}{3}] \qquad (32)$$

$$I_{yy} = 2MR^2[\sin^2(2\pi ft) - \frac{1}{3}] \qquad (33)$$

$$I_{xy} = I_{yx} = 2MR^2[\cos(2\pi ft)\sin(2\pi ft)] \qquad (34)$$

From equation no (30) and (32) – (34) it then follows that

$$h_{xx} = -h_{yy} = h\cos(4\pi ft) \qquad (35)$$

and $$h_{xy} = h_{yx} = -h\sin(4\pi ft) \qquad (36)$$

where the amplitude term h is given by

$$h = \frac{32\pi^2 GMR^2 f^2}{c^4 r} \qquad (37)$$

The binary system emits gravitational waves at twice the orbital frequency of the neutron stars. Suppose we take *M* equal to the Chandrasekhar mass, $M \sim 1.4M_{solar} = 2.78 \times 10^{30} kg$. We can then evaluate the constants in(37) and express *h* in more convenient units as

$$h = 2.3 \times 10^{-28} \frac{R^2[\mathrm{km}]f^2[\mathrm{Hz}]}{r[\mathrm{Mpc}]} \qquad (38)$$

If we take $R = 20km$, say, $f = 1000Hz$ (which is approximately the orbital frequency that Newtonian gravity would predict) and $r = 15Mpc$ (corresponding to a binary system in e.g. the Virgo cluster), then we find that

$h \sim 6 \times 10^{-21}$. Thus we see that the signal produced by a typical gravitational wave source places extreme demands upon detector technology.

Concluding Remarks

The direct detection of GWs by LIGO initiates a new era of GW astronomy and GW cosmology. The GW physics is not only related to gravitational physics, but also closely related to particle physics, cosmology and astrophysics. The GWs provide us a new powerful tool to reveal various secrets of the nature. Indeed, a lot of relevant papers have appeared since the announcement of the direct detection of GWs. In this paper, we have briefly introduced three kinds of GW sources and relevant physics. The GWs are produced during inflation and preheating in the early Universe, from cosmic PTs and dynamics of compact binary systems, (Binary Black Holes and Binary Neutron Stars, etc,…) respectively. We also have discussed in a simple way the GWs as standard siren in the evolution of the Universe.

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References

A. Einstein, Approximative Integration of the Field Equations of Gravitation, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) 1916 (1916) 688–696.

A. H. Guth, (1981), “The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems”, Phys. Rev. D23, 347–356.

A. D. Linde, (1982), “A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems”, Phys. Lett. B108, 389–393.

B. P. Abbott et al., (2016) Virgo, LIGO Scientific collaboration “Observation of Gravitational Waves from a Binary Black Hole Merger”, Phys. Rev. Lett.116,061102, [1602.03837].

B. P. Abbott et al., Virgo, LIGO Scientific collaborationGW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence, Phys. Rev. Lett. 116 (2016) 241103, [1606.04855].

H. Bondi, Plane gravitational waves in general relativity, Nature 179 (1957) 1072–1073.

P. R. Saulson, Josh Goldberg and the physical reality of gravitational waves, Gen. Rel. Grav. 43 (2011) 3289–3299.