J. Myanmar Acad. Arts Sci. 2019 **Vol. XVII. No.2B**

AN ANALYSIS OF NUMERICAL HYDRODYNAMICS FOR SPHERICALLY SYMMETRIC SPACETIMES

Yee May Thwin[1], Zaw Myint[2]

Abstract

The interesting features of numerical hydrodynamics for spherically symmetric spacetimes has been presented in the context of general relativity. Some distinct results are visualized and physical interpretations have been given.

Keywords: numerical hydrodynamics, spherically symmetric spacetimes

Introduction

The description of important areas of modern astronomy, such as high-energy astrophysics or gravitational wave astronomy requires general relativity. Einstein's theory of gravitation plays a major role in astrophysical scenarios involving compact objects such as neutron stars and black holes. High-energy radiation is often emitted in regions of strong gravitational fields near such compact objects. The production of relativistic radio jets in active galactic nuclei, explained by either hydrodynamic or electromagnetic mechanisms, involves rotating supermassive black holes. The discovery kHz quasi-periodic oscillations in low-mass X-ray binaries extended the frequency range over which these oscillations occur into timescales associated with the relativistic, innermost regions of accretion disks. A relativistic description is also necessary in scenarios involving explosive collapse of very massive stars to black hole, or during the last phases of the coalescence and merge of neutron star binaries and neutron-star-black-hole binaries. These catastrophic events are believed to occur at the central engine of the most highly energetic events in nature, gamma-ray bursts (GRBs). Astronomers have long been scrutinizing these systems using the complete frequency range of the electromagnetic spectrum. Nowadays, they are the main targets of ground-based laser interferometers of gravitational radiation. The direct detection of these elusive ripples in the curvature of space time, and the wealth of new information that could be extracted from them, is one of the driving motivations of present-day research in relativistic astrophysics.

1. Lecturer, Department of Physics , University of Yangon, Myanmar
2. Assistant Lecturer, Department of Physics, East Yangon University, Myanmar

An accurate description of relativistic flows with strong shocks is nowadays demanded for the study of a large number of important problems in physics and astrophysics. Ultra relativistic flows are found not only in extragalactic jets (Begelman, Blandford, & Rees 1984; see also Marti et al. 1997 for an up-to-date bibliography) but also in high-energy heavy-ion collisions (Clare & Strottman 1986). General relativistic effects caused by the presence of strong gravitational fields appear connected to extremely fast flows in different astrophysical scenarios, e.g., accreting compact objects, stellar collapse, and coalescing compact binaries (Shapiro & Teukolsky 1983; Thorne 1987; Bonazzola & Marck 1994). In recent years, much effort has been addressed to developing accurate numerical algorithms able to solve the equations of general relativistic hydrodynamics in the extreme conditions described above. The main conclusion that has emerged is that modern algorithms exploiting the hyperbolic (and conservative) character of the system of equations are by far more accurate at describing relativistic flows than traditional finite-difference upwind techniques with artificial viscosity (introduced by Wilson 1972,1979). Wilsons work marked further developments for the integration of the relativistic hydrodynamics system of equations(Piran 1983; Stark & Piran 1987; Nakamura et al. 1980; Nakamura 1981; Nakamura & Sato 1982; Centrella & Wilson 1984;Hawley, Smarr, & Wilson 1984a, 1984b; Evans 1986). However, the procedure seems to break down for relativistic flows with high Lorentz factors, for which large numerical inaccuracies and instabilities are obtained (Norman & Winkler 1986). Totake advantage of the conservation properties of the system, modern algorithms are written in conservation form, in the sense that the variation of the mean values of the conserved quantities within the numerical cells is given, in the absence of sources, by the fluxes across the cell boundaries. Furthermore, the hyperbolic character of the system of equations allows one to obtain these fluxes from solutions of discontinuous initial problems (i.e., Riemann problems) between neighboring numerical cells. In this way, physical discontinuities appearing in the flow are treated consistently (the shock-capturing property).

The use of Riemann solutions in numerical codes comes from the idea of Godunov (1959), who first introduced them in classical fluid dynamics, but it was not until the late seventies when, thanks to the development of new cell-reconstruction procedures (van Leer 1979; Colella & Woodward 1984a, 1984b; Marquina 1994), high-resolution shock-capturing (HRSC)techniques were recognized as the most effective way to describe complex flows accurately. Since then, efficient Riemann solvers based on exact or approximate solutions of the initial-value problem have been developed. A general approach, followed here, is to obtain the fluxes from the solution of a linearized form of the original system of equations (the local characteristic approach). This solution can be obtained exactly by writing the system in terms of the so-called characteristic variables. In terms of these variables, which are obtained by projection of the original variables onto the right eigenvectors of the Jacobian matrices, the system decouples into a set of scalar advection equations, the eigenvalues of the Jacobian matrices being the advection velocities (characteristic speeds). Intrinsic to this approach is the spectral decomposition of the Jacobian matrices of the partial derivative system of equations.

Equations of General Relativistic Hydrodynamics as a System of Conservation Laws

The equations that describe the evolution of a relativistic fluid are local conservation laws: the local conservation of baryon number,

$$\nabla . J = 0$$

and the local conservation of energy-momentum,

$$\nabla . T = 0$$

where $\nabla .$ stands for the covariant divergence. If $\{\partial_t, \partial_i\}$define the coordinate basis of 4-vectors that are tangents to the corresponding coordinate curves, then the vector ***J***-the current of rest mass - and the bilinear form T - the energy-momentum tensor have the components

$$J^{\mu} = \rho u^{\mu} \,,\ J^{\mu\nu} = \rho h u^{\mu} u^{\nu} + p g^{\mu\nu}$$

ρ being the rest-mass density, p the pressure, and h the specific enthalpy, defined by $\rho\, h = 1 + \varepsilon + \dfrac{p}{\rho}$ where ε is the specific internal energy. Here u^{μ} is the

4-velocity of the fluid, and $g_{\mu\nu}$ defines the metric of the spacetime *M*in which the fluid evolves.

Throughout this paper, Greek (Latin) indices run from 0 to3 (1 to 3)- or, alternatively, they stand for the general coordinates $\{t, x, y, z\}(\{x, y, z\})$-and geometrized units are used(c=G=1). An equation of state p=p (ρ,ε), as usual, closes the system.

A very important quantity derived from the equation of state is the local sound velocity, c_s:

$$hc_s^2 = \chi + \left(\frac{p}{\rho^2}\right)\kappa,$$

with $\chi = \frac{\partial p}{\partial \rho}|_{\varepsilon}$and $\kappa = \frac{\partial p}{\partial \varepsilon}|_{\rho}$. Let $\mathcal{M}$ be a general spacetime, described by the four-dimensional metric tensor $g_{\mu\nu}$.

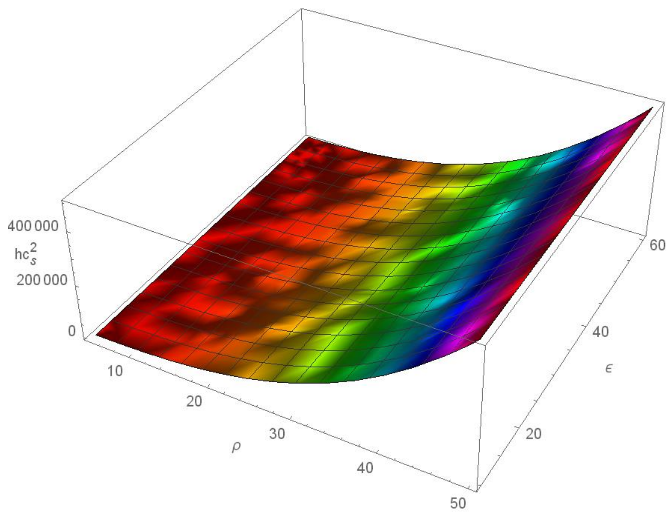

Figure 1: The variation of the local sound velocity c_s with ρ and ε

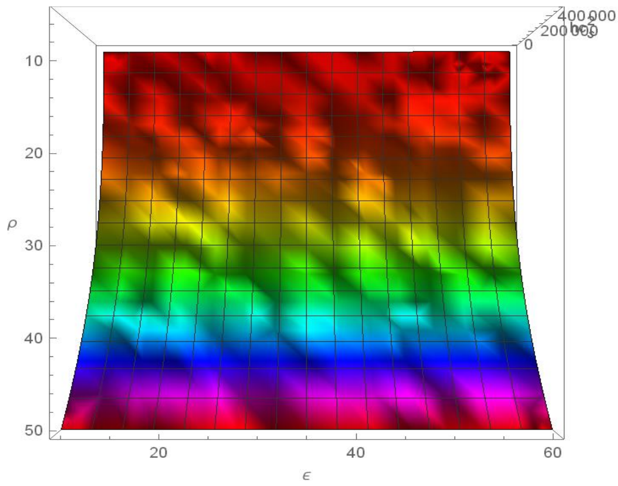

Figure 2: The variation of the local sound velocity c_s with ρ and ε

According to the {3+1} formalism the metric is split into the objects $\alpha(lapse)$, β^i (shift), and γ_{ij}, keepingthe line element in the form

$$ds^2 = -(\alpha^2 - \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ij} + dx^i dx^j$$

If $\boldsymbol{n}$ is a unit timelike vector field normal to the spacelike hyper surfaces Σ_t (t=const), then, by definition of α and β^i,

$$\partial_t = \alpha n + \beta^i \partial_i$$

with $n \, . \, \partial_i = 0$ for all i.

Observers $\mathcal{O}_E$ at rest in the slice Σ_t, i.e., those having $\boldsymbol{n}$ as 4-velocity measure the following velocity of the fluid:

$$\upsilon_i = \frac{u \, . \, \partial_i}{-u \, . \, n},$$

where the contravariant components $\upsilon^i = \gamma^{ij} \, \upsilon_j$ are

$$\upsilon^i = \frac{u^i}{\alpha u^t} + \frac{\beta^i}{\alpha}$$

and the denominator. In equation is the Lorentz factor $W \equiv -u\,.\,n = \alpha\,u^t$ which satisfies $W = (1 - \upsilon^2)^{-1/2}$ with $\gamma_{i\,j}\,\upsilon^i \upsilon^j$.

Let us define a basis adapted to the observer $\mathcal{O}_E$

$$e_{(\mu)} = \{n, \partial_{\mathrm{i}}\},$$

and the following five 4-vectors $\mathcal{D}_{(A)}$:

$$D_{(A)} = \{T(e_{(\gamma)}, .), J\}, \quad \mathrm{A}=0,\ldots,4.$$

Hence the above system of equations (1) and (2) can be written

$$\nabla .\, D_{(A)} = s_{(A)}$$

The five quantities $s_{(A)}$ on the right-hand side of sources are equation the

$$s_{(A)} = T^{\mu\nu} \nabla_{\mu} e$$

$$s_{(A)} = \{T^{\mu\nu} \overleftarrow{\nabla}_{\mu} e_{(\gamma)\nu}, 0\}, \quad \nabla_{\mu} e_{(\mu)\nu} = \frac{\partial e_{(\gamma)\nu}}{\partial \chi^{\mu}} - \Gamma^{\delta}_{\nu\mu}\, e_{(\gamma)\delta} \;,$$

where the quantities $\Gamma^{\mu}_{\beta\nu}$ are the Christoffel symbols and

$$e_{(0)\nu} = -\alpha \delta_{0\nu}, \qquad e_{(k)\nu} = g_{k\nu} = (\beta_k, \gamma_{kj}) \;.$$

Taking into account those quantities that are directly measured by $\mathcal{O}_E$ i.e., the rest-mass density (D), the momentum density in the j-direction (S_j), and the total energy density (E), we can display them in terms of the primitive variables

$$w = (\rho\,, \upsilon_{\mathrm{i}}, \varepsilon)^{\mathrm{T}}$$

by the following relations :

$$D \equiv -J.n = \rho\,\mathrm{W},\; S_j \equiv -T(n, e_{(j)}) = \rho\, h W^2 \upsilon_{\mathrm{j}},\; E \equiv -T(n,n) = \rho\, h W^2 - \mathrm{p}\,.$$

Putting together all the above relations, the fundamental system to be considered for numerical applications is

$$\frac{1}{\sqrt{-g}}\left[\frac{\partial\sqrt{\gamma}\,F^{0}(w)}{\partial x^{0}}+\frac{\partial\sqrt{-g}\,F^{i}(w)}{\partial x^{i}}\right]=s(w),$$

where the quantities $F^{\alpha}(w)$ are

$$F^{0}(w)=(D,S_{j},\tau)^{T}$$

$$F^{i}(w)=\left[D\left(\upsilon^{i}-\frac{\beta^{\mathrm{i}}}{\alpha}\right),S_{j}\left(\upsilon^{i}-\frac{\beta^{i}}{\alpha}\right)+p\delta^{i}_{j},\tau\left(\upsilon^{i}-\frac{\beta^{i}}{\alpha}\right)\right]^{T}+p\upsilon^{i}$$

and the corresponding sources $s(w)$ are

$$s(w)=\left[0,\ T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial\chi^{\mu}}-\Gamma^{\delta}_{\nu\mu}g_{\delta j}\right),\alpha\left(T^{\mu 0}\frac{\partial\ln\alpha}{\partial\chi^{\mu}}-T^{\mu\nu}\Gamma^{0}_{\nu\mu}\right)\right]^{T},$$

τ being $\tau\equiv E-D$; that is, the total energy density minus the rest-mass density, $g\equiv\det g_{\mu\nu}$ is such that

$$\sqrt{-g}\equiv\alpha\sqrt{\gamma}\ ,\qquad \gamma\equiv\det\gamma_{ij}\ ,$$

and "det" stands for the determinant of the corresponding matrix. The quantities F^{α} have been expressed in terms of the physical quantities measured by $\mathcal{O}_{E}$ which are the conserved variables. It is worthwhile, for numerical purposes, to point out, that the sources do not contain any differential operator acting on the components of w, which is a fundamental condition for preserving, numerically, the hyperbolic character of the system.

Concluding remarks

It has been presented that the most fundamental elements of the mathematical structure of multidimensional general relativistic hydrodynamics as a hyperbolic system of conservation laws. The analysis acquires an outstanding relevance in the context of numerical relativistic astrophysics. This study has been carried out in terms of the {3+1} formalism, which is well suited for the solution of the Einstein field equations. The spectral decomposition of the Jacobian matrices of the system, necessary to build up a Riemann solver and for taking advantage of the local characteristic approach, have been explicitly derived.

Acknowledgements

I would like to thank Dr Khin Khin Win, Professor and Head, Department of Physics, University of Yangon, for her kind permission and encouragements to carry out this work.

I would like to thank my respected Dr Aye Aye Thant, Professor, Department of Physics, University of Yangon, for her encouragements to carry out this work.

Special thanks are due to Professor, Dr Thant Zin Naing, Retired Pro-rector(Admin), International Theravāda Buddhist Missionary University, for his valuable guidance and helpful advice to carry out this work.

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