SIMPLE ASTROPHYSICAL SIMULATIONS FOR COLLISIONLESS STELLAR SYSTEMS

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Abstract

Simple astrophysical simulations for stellar systems which are assumed to be collision less have been studied using N-body simulation techniques. Physical entities such as potential-density pair profiles for the dark halo and stellar bulge have been simulated and some interesting remarks are given.

Keywords: *collision less stellar systems, N-body simulation, stellar disk, dark halo, stellar budge*

Introduction

The underlying dynamics relevant in the astrophysical context for a system of N particles interacting gravitationally is typically Newton's law plus, in case, an external potential field (see however below for a discussion of N-body simulations in general relativity). The force F_i acting on particle i of mass m_i is:

$$\vec{F}_i = -\sum_{j\neq i} \frac{Gm_i m_j(\vec{r}_i - \vec{r}_j)}{\left|\vec{r}_i - \vec{r}_j\right|^3} - \vec{\nabla}.\phi_{ext}(\vec{r}_i) \qquad (1)$$

Where $G = 6.673\times10^{-11} m^3 kg^{-1} s^{-2}$ is the gravitational constant, and ϕ_{ext} is the external potential. The problem is thus a set of non-linear second order ordinary differential equations relating the acceleration $\frac{\partial^2 \vec{r}_i}{\partial t^2} = \frac{\vec{F}_i}{m_i}$ with the position of all the particles in the system. Once a set of initial condition is specified (for example the initial positions r_i and velocities $\bar{v}_i \equiv \frac{\partial r_i}{\partial t}$ of all particles) it exists a unique solution, analytical only for up to two bodies, while larger N require numerical integration (e.g. see Press et al. 2007). However special care must be employed to ensure both accuracy and efficiency. In fact, the gravitational force (eq.1) presents a singularity when

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the distance of two particles approaches 0, which can lead to arbitrarily large relative velocities. In depend on the specific choice of initial conditions. In contrast, all singularities in linear ordinary differential equations are independent of initial conditions and thus easier to treat. Therefore constant time step methods are unable to guarantee a given accuracy in the case of gravitational dynamics and lead to unphysical accelerations during close encounters, which in turn may create unbound stars.

A shared adaptive time step scheme can correctly follow a close encounter, but the price is paid in terms of efficiency as all the other particles of the system are evolved on the timescale of the encounter, which may be several orders of magnitude smaller than the global timescale, resulting essentially in a freezing of the system. The singularity may be avoided by introducing a smoothing length in Eq. 1 (e.g. see Aarseth 1963), that is by modifying the gravitational interaction at small scales. For example:

$$\vec{F}_i = -\sum_{j\neq i} \frac{G m_i m_j (\vec{r}_i - \vec{r}_j)}{\{\left|\vec{r}_i - \vec{r}_j\right|^2 + \in^2\}^{3/2}} \qquad (2)$$

where $\epsilon > 0$ is the softening, or smoothing length, that is a typical distance below which the gravitational interaction is suppressed. To minimize the force errors and the global impact of the softening for distances larger than $\in$, finite size kernels that ensure continuous derivatives of the force may be employed (e.g., see Dehnen 2001). This strategy effectively suppresses binary formation and strong gravitational interactions, but at the price of altering the dynamics of the system.

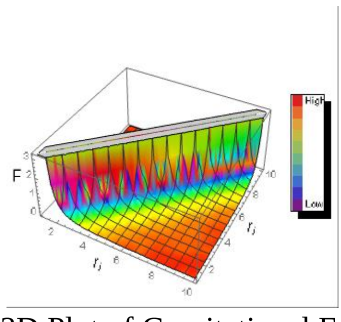

Figure 1: The 3D Plot of Gravitational Force against distance

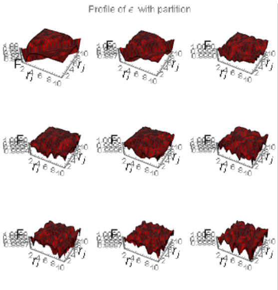

Figure 2: Snapshot - profiles of the evolution of $\in$ for Gravitational force F

Timescales, Equilibrium and Collisionality

A system of N particles interacting gravitationally with total mass M and a reference dimension R (for example the radius containing half of the total mass) reaches a dynamic equilibrium state on a timescale comparable to a few times the typical time (T_{cr}) needed for a particle to cross the system ($T_{cr} \approx 1/\sqrt{GM/R^3}$). This is the response time needed to settle down to virial equilibrium, that is 2K/|W|=1, where K is the kinetic energy of the system $K = 1/2\sum_{i=1,N} m_i |\bar{v}_i|^2$, and W is its potential energy: $W = -1/2\sum_{i\neq j} G m_i m_j \Big/ |\bar{r}_i - \bar{r}_j|^2$ (assuming no external field). If the system is initially out of equilibrium, this is reached through mixing in phase space due to fluctuations of the gravitational potential, a process called violent relaxation.

Once the system is in dynamic equilibrium a long term evolution is possible, driven by two-body relaxation. Energy is slowly exchanged between particles and the system tends to evolve toward thermodynamic equilibrium and energy equipartition. The timescale (T_{rel}) for this process depends on the number of particles and on the geometry of the system: $T_{rel} \propto N/\log(0.11N)T_{cr}$. N-body systems such as galaxies and dark matter halos have a relaxation time much longer than the life of the Universe and are thus considered collision less systems. Smaller systems, such as globular and open clusters, are instead collisional, as the relaxation time is shorter than their age. Two body relaxation is also suppressed when one particle in the system dominates the gravitational potential, such as in the case of solar system dynamics, where planets are essentially quasi-test particles. Close encounters between three or more particles not only contribute to energy exchange, but can also lead to the formation of bound subsystems (mainly binaries). The formation and evolution of a binary population is best followed through direct, unsoftened, N-body techniques. A self-gravitating N-body system made of single particles has a negative specific heat, that is it increases its kinetic energy as a result of energy losses. This is a consequence of the virial theorem and qualitatively it is analogous to the acceleration of a Earth artificial satellite in presence of atmospheric drag. A negative specific heat system is thermodynamically unstable and over the two body relaxation timescale it evolves toward a gravothermal collapse, creating a core-halo structure, where the core progressively increases its concentration, fueling an overall halo expansion. The collapse is eventually halted once three body interactions lead to the formation of binaries. The so called core collapsed globular clusters" are considered to be formed as a result of this mechanism.

Methods for cosmological N-body simulations

Cosmological N-body simulations play an important role in modern cosmology by providing vital information regarding the evolution of the dark matter: its clustering and motion, about properties of dark matter halos. The simulations are instrumental for the transition of the theoretical cosmology from an inspiring but speculative part of astronomy to the modern precision cosmology. In spite of more than 50 years of development, N-body methods are still a thriving field with invention of more powerful methods providing more accurate theoretical predictions.

Dark matter is important component of universe. All observational evidence indicates that it dominates dynamics of normal and dwarf galaxies clusters and groups of galaxies. At high red shifts it provided the force that drove the formation of first galaxies and quasars. The observed large filaments and giant voids all can be understood and explained if we combine the dynamics of dark matter with the predictions of the inflation model on the spectrum of primordial fluctuations.

The dark matter is likely made of particles that other than the gravity force do mot couple with the other matter. There may be some channel of interactions between dark matter particles resulting in annihilation and production of normal particles. However, even if present (no observational evidence so far), this channel is weak and the dark matter is (mostly) preserved over the evolution of the Universe.

How this dark matter evolves and how it forms different structures and objects was an active field of research for a very long time. The first (Somewhat) realistic N-body simulation collapse of a cloud of 300 self-interacting particles was done by P.J.E. Peebles Peebles (1970). Remember that at that time of the dawn of cosmology, there was no dark matter, the hot gas x-ray in clusters had not been yet discovered (it was discovered in 1971), there were no voids or superclusters. So the first N-body simulation had indicated that the force of gravity alone may be responsible for the formation of clusters of galaxies, which was a big step forward. It also discovered a problem the density profile in the model was not right: too steep. The Solution for this problem was continuous mass accretion on the forming cluster instead of a one-time event of collapse (Gunn & Gott 1972)

From that moment the simulations took off. Larger and larger numbers of particles were used as new codes and new computers became available. For some time it looked almost like a sport: whose simulation has more "muscle". The pace has slowed down in recent years mostly because it became more difficult to analyse the simulations and to make the results accessible to the larger community. Development of numerical methods was crucial for advances in N-body simulations. At the beginning direct summation technique was used to run the simulations (Peebles1970;White 1976; Aarseth et al. 1979). At that time slower processors, no parallel computing it was difficult to make simulations with more than just a few thousand particles.

The main motivation at that time was to develop new computational methods. The number of operations in the direct summation method scales as $\propto N^2$, where N is the number of particles. So, one quickly ran out of available cpu. However, now the situation is different: processors are much faster and the number of cores on a workstation can be significant. A simulation with $N = 10^5 - 10^6$ is relatively fast(from few hours to few days). Such simulations can be very useful for testing different ideas and for small runs. It is also very easy to modify the code because everything is very transparent. For example, one can add external tidal force or modify the law of gravity. It is also a great tool for training students: a simple parallel pair-wise summation code can be written in few hours. Particle-Mesh method a big step forward with cpu scaling $\propto N^2$ However, it requires a large 3D mesh for computation of the gravitational potential. The size of a cell in that mesh defines the force resolution, and, if one needs better resolution, the number of cells should be increased. As the result, one may run out of available computer memory. Still, the PM method is very fast and is easy to implement.

Cosmological N-body problem: main equations

In order to derive equations for the cosmological N-body problem, one can start with the equations of general relativity and derive equations of motion of self-gravitating nonrelativistic particles in the expanding Universe. For the case of nonrelativistic matter and the weak-field limit, we simply arrive at the Newtonian equations. There are some limitations with this approach: we cannot treat relativistic particles and we neglect time needed for gravitational perturbations to travel from one point to another effectively treating changes in the gravitational potential as instantaneous. However, these effects are not significant for most applications: velocities are typically well below relativistic and effects of the finite time of gravitational perturbations are small. We start with definitions, proper position r and comoving coordinates x are related:

$$r(x,t) = a(t)x(t) \tag{3}$$

where a(t) is the expansion factor. Differentiating eq.(3) overtime, we get velocities:

$$v(x,t) \equiv \dot{r} = a\dot{x} + \dot{a}x = Hr + v_{pec} \tag{4}$$

Here $v_{pec} = a\dot{x}$ is the peculiar velocity and $H = \dot{a}/a$ is the Hubble constant. It is also useful to introduce the specific momentum defined as $p \equiv a^2\dot{x} = av_{pec}$

In cosmology we deal with a rather specific case of the N-body problem. Here discreteness of matter can be neglected. In general this is not the case with the two-body effects gradually accumulating over time. Systems studied in cosmology such as the nonlinear evolution of dark matter clustering do not suffer from the two-body scattering and can be treated using the collision less Boltzmann equation paired with the Poisson equation for the gravitational potential. In the comoving coordinates the Boltzmann equation describing the evolution of the distribution function f (x, p, t) can be written as:

$$\frac{\partial f}{\partial t} + x\frac{\partial f}{\partial x} - \nabla\phi\frac{\partial f}{\partial p} = 0 \tag{5}$$

where peculiar gravitational potential $\phi(x)$ is related with the normal gravitational potential Φ as $\Phi = 2\pi G\rho_b r^2/3 + \phi$ where the first term is the potential of the background (constant over space) density field ρ_b and the second term is the deviation from the background. Changing coordinates from proper r to comoving x we can write the Poisson equation as:

$$\nabla^2\phi = 4\pi G a^2(\rho(x) - \rho_b) = 4\pi G\frac{\Omega_0\rho_{cr0}}{a}\delta_{dm}(x,t) \tag{6}$$

Here $\delta_{dm} \equiv (\rho_{dm}(x,t) - \langle\rho_{dm}\rangle)/\langle\rho_{dm}\rangle$ is the dark matter density contrast. Factors Ω_0 and $\rho_{cr,0}$ are the average matter(dark matter plus baryons) density in the units of the critical density and the critical density all taken at the present moment a = 1.Note that the right hand side of eq.(6) may have a positive or negative sign. This is unusual considering that in a normal Poisson equation the density is always positive. The negative sign of the density term in eq.(6) happens in locations where the density is below the average density of the Universe. While there are no real negative densities in the Poisson equation, the regions with the negative r.h.s. of eq.(6) in comoving coordinates act as if there are. For example, in these regions the peculiar gravitational acceleration points away from the centre of an under dense region resulting in matter being pushed away from the centre. This explains why over time voids (large under dense regions) observed in the large-scale distribution of the dark matter become bigger and more spherical.

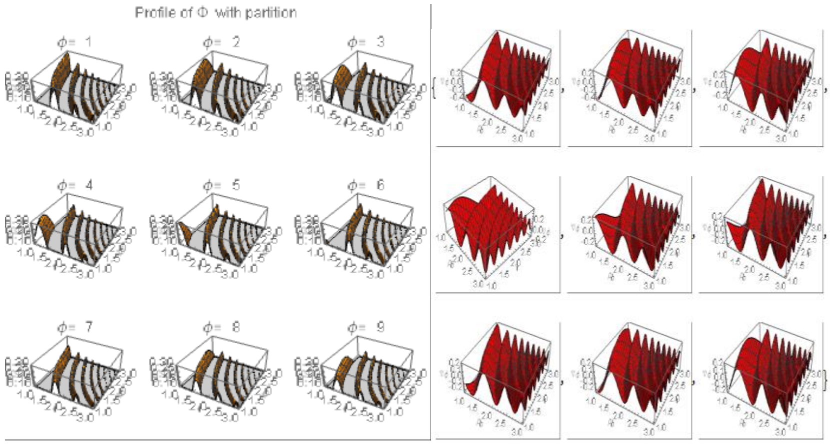

Figure 3: Snapshot -profiles of the evolution of ϕ for Gravitational Potential

The collision less Boltzmann equation eq.(5) is a linear first order partial differential equation in the 7-dimensionalspace (x,p, t). It has a formal solution in the form of characteristics: a set of curves that cover the whole space. The characteristics do not intersect and do not touch each other. Along each characteristic the value of the distribution function is preserved. In other words, if at some initial moment t_i we have coordinate x_i, momentum p_i, and phase-space density f_i, then at any later moment t along the characteristic we have $f(x, p, t) = f_i(x_i, p_i, t_i)$. Equations of the characteristics, the Piosson equation, and the Friedmann equation can be written as follows:

$$\frac{dx}{da} = \frac{P}{a^3 H}, \frac{dp}{da} = -\frac{\nabla\phi}{aH} \quad (7)$$

$$\nabla^2 \varphi = \frac{3}{2}\frac{H_0^2 \Omega_0 \phi \delta_{dm}}{a} \quad (8)$$

$$H^2 = H_0^2 \left(\frac{\Omega_0}{a^3} + \Omega_{\Lambda,0}\right) \; \Omega_0 + \Omega_{\Lambda 0} = 1 \quad (9)$$

Here we specifically assumed a flat cosmological model with the cosmological constant characterized by the density parameter, $\Omega_{\Lambda 0}$ at red shift z = 0.

There are numerical factors in eqs.(7–8) that obscure the fact that the equations of characteristics are nothing but the equations of motion of particles under the force of gravity. These equations are almost the equations of the N-body problem in the comoving coordinates. However, there are differences. Characteristics cover the whole phase-space which we cannot do in simulations that use a finite number of particles. Instead, we approximate the phase-space by placing particles at some positions and giving them initial momenta. How exactly we place the particles depends on the problem to be solved. For example, if a large simulation volume is expected to be resolved everywhere with the same accuracy, then particles should be nearly homogeneously distributed initially and have the same mass. If instead a small region should be resolved with a higher resolution than its environment, than we place lots of small particles in the region and cover the rest of the volume with few large particles. Because we intend to produce an approximate solution for the continuous distribution of matter in space as described by the Boltzmann-Poisson equations, we may not even think that we solve the N-body problem an ensemble of point masses moving under the force of gravity. For example, at the initial moment the volume of a simulation may be covered by many small non-overlapping cubes (not points).Then each cube is treated as a massive particle with some size, mass, and momentum. So, instead of N point masses we have N small cubes. This is definitely a better approximation for the reality. Indeed, these types of approximations are used in many simulations. For example, in Particle-Mesh(PM) simulations dark matter particles are small cubes with constant density and size. In Adaptive Mesh Refinement (AMR) codes particles are also cubes with the size of the cube decreasing in regions with better force resolution. The last clarification is related to the baryons. In order to treat the baryons properly, we need to include equations of hydrodynamics and add gas density to the Poisson equation. We clearly do not do it in N-body simulations. Still, we cannot ignore baryons. They constitute a significant fraction of mass in the Universe. If we neglect baryons, there will be numerous defects. For example, they growth rate of fluctuations even on large scales will be wrong and virial masses will not be correct. In cosmological N-body simulations we assume that all the mass-dark matter and baryons is in particles and each particle represents both dark matter and baryons with the ratio of the two being equal to the cosmological average ratio.

Simple N-body problem: pair-wise summation

We start discussion of numerical techniques with a very simple case: forces are estimated by summing up all contributions from all particles and with every particle moving with the same time-step. The computational cost is dominated by the force calculations that scale as N^2, where Nis the number of particles in the simulation. Because of the steep scaling, the computational cost of a simulation starts to be prohibitively too large for $N \geq 10^6$ However, simulations with a few hundred thousand particles are fast, and there are numerous interesting cases that can be addressed with $N \rangle 10^6$ particles. Examples include major-mergers of dark matter halos, collisions of two elliptical galaxies, and tidal stripping and destruction of a dwarf spheroidal satellite galaxy moving in the potential of the Milky Way galaxy. In these cases it is convenient to use proper, not commoving coordinates.

The problem that we try to solve numerically is the following. For given coordinates r_{init} and velocities v_{init} of N massive particles at moment $t = t_{init}$ find their velocitiesv and coordinates r at the next moment $t = t_{next}$ assuming that the particles interact only through the Newtonian force of gravity. If r_i and m_i are the coordinates and masses of the particles, then the equations of motion are:

$$\frac{d^2 r_i}{dt^2} = -G \sum_{j \neq i, i \neq j} \frac{m_j(\vec{r}_i - \vec{r}_j)}{\left|\vec{r}_i - \vec{r}_j\right|^3} \tag{10}$$

Where G is the gravitational constant. Two steps should betaken before we start solving equations (8) numerically First, we introduce force softening: we make the force weaker (“softer”) at small distances to avoid very large accelerations, when two particles collide or come very close to each other. This makes numerical integration schemes stable. Another reason for softening the force at small distances is that in cosmological environments, when one deals with galaxies, clusters of galaxies, or the large-scale structure, effects of close collisions between individual particles are very small and can be neglected. In other words, the force acting on a particle is dominated by the cumulative contribution of all particles, not by a few close individual companions. There are different ways of introducing the force softening. For mesh-based codes, the softening is defined by the size of cell elements. For TREE codes the softening is introduced by assuming a particular kernel, and it is different for different implementations. The simplest and often used method is called the Plummer softening. It replaces the distance between particles $\Delta r_{ij} = \left| r_i - r_j \right|$ in eq.(10)with the expression $(\Delta r_{ij} + \in^2)^{1/2}$, where $\in$ is the softening parameter.

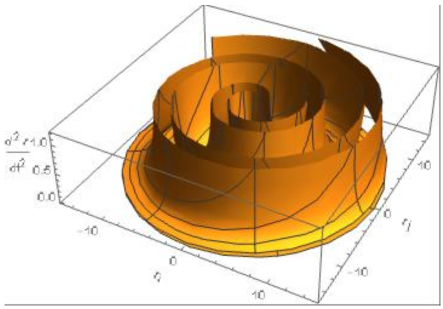

Figure 4: The Revolution Plot of Acceleration against distance

Second, we need to introduce new variables to avoid dealing with too large or too small physical units of a real problem. This can be done in a number of ways. For mesh-based codes, the size of the largest resolution element and the Hubble velocity across the element give scales of distance and velocity. Here we use more traditional scaling. Suppose M and R are scales of mass and distances. These can be defined by a particular physical problem. For example, for simulations of an isolated galaxy M and R canbe the total mass and the initial radius. It really does not matter what M and R are. The scale of time t_0 is chosen $t_0 = (GM/R^3)^{-1/2}$. Using M, R, and t_0 we can change the physical variables r_i, v_i, m_i into dimensionless variables using the following relations

$$\vec{r}_i = \tilde{r}\vec{R}, v_i = v_i \frac{\vec{R}}{t_0}, m_i = \tilde{m}_i M, t = \tilde{t}\, t_0 \tag{11}$$

We now change the variables in eq. (10) and use the Plummer softening

$$\tilde{g}_i = -\sum_{j=1} \frac{\tilde{m}_j(\tilde{r}_i - \tilde{r}_j)}{(\Delta r_{ij} + \tilde{\epsilon}^2)^{3/2}} \quad , \frac{d\tilde{v}_i}{d\tilde{t}} = \tilde{g}_i , \frac{d\tilde{r}_i}{d\tilde{t}} = \tilde{v}_i \qquad (12)$$

Where, $\tilde{g}_i$ is the dimensionless gravitational acceleration. Note that these equations look exactly as eq (10), if we formally set G=1 and $\epsilon = 0$.

The simplest, but not the best, method to derive the Green functions is to consider $\phi_{i,j,k}$ and $\rho_{i,j,k}$ as amplitudes of the Fourier components of the gravitational potential in the computational volume and then to differentiate the Fourier harmonics analytically. This gives

$$G_0(k) = -\frac{1}{k_x^2 + k_y^2 + k_z^2} = -\left(\frac{L}{2\pi}\right)^2 \frac{1}{i^2 + j^2 + k^2} \qquad (13)$$

where $(k_x, k_y, k_z) = (2\pi/L)(i, j, k)$ are components of the wave-vector in physical units. A better way of solving the Poisson equation is to start with the finite-difference approximation of the Laplacian∇^2. Here we use a the second order Taylor expansion for the spacial derivatives:

$$\begin{aligned} \nabla^2\phi &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \\ &\approx [\phi_{\iota+1,j,\kappa} - 2\phi_{\iota,j,\kappa} + \phi_{\iota-1,j,\kappa} + \phi_{\iota,j+1,\kappa} - 2\phi_{\iota,j,\kappa} + \phi_{\iota,j-1,\kappa} + \phi_{\iota,j,\kappa+1} - 2\phi_{\iota,j,\kappa} + \phi_{\iota,j,\kappa-1} \end{aligned} \qquad (14)$$

This approximation leads to a large system of linear algebraic equations: $A\phi = 4\pi G\rho$, where ρ is the vector on the right hand side, ϕ is the solution, and A is the matrix of coefficients. The solution of this matrix equation can be found by applying the Fourier Transformation. This provides another approximation for the Green functions:

$$G_1(k) = \frac{\Delta x^2}{2} \times \left[\cos\left(\frac{2\pi i}{N_{grid}}\right) + \cos\left(\frac{2\pi j}{N_{grid}}\right) + \cos\left(\frac{2\pi k}{N_{grid}}\right) - 3\right]^{-1} \qquad (15)$$

For small (i, j, k) eq.(14) gives the same results as $\phi = 4\pi G\rho$. However, at (i, j, k) close to N_{grid} the finite-difference scheme G_1 provides less suppression for high-frequency harmonics and thus gives a stronger and more accurate force at distances close to the grid spacing Δx. The computer memory puts constraints on the PM method because the method requires a large 3-dimensionalmesh of size N^3_{grid} while the force resolution increases only as the first power of N_{grid} : $\Delta x = L/N_{grid}$, where L is length of the computational box. As we start to increase the resolution, we quickly run of the computer memory.

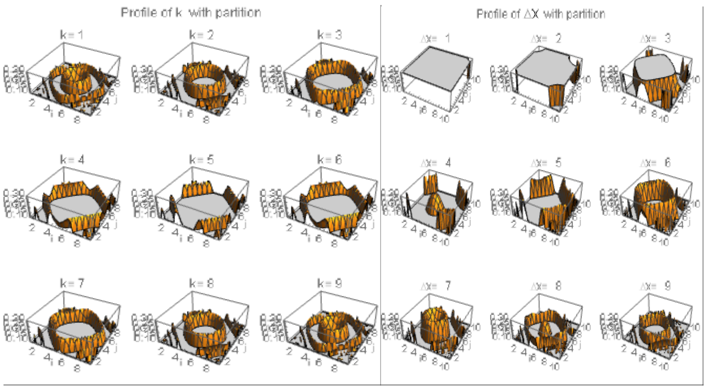

Figure 5: Snapshot -profiles of the evolution of k and Δ x for Green Functions

Concluding Remarks

An alternative approach to implement the numerical solution of the N-body problem is to make use of the simple codes of Mathematica or MaTLab. For direct simulations this approach can be very effective, thanks to the fact that the bottle neck of computation is just the evaluation of the gravitational force, which has a very simple expression. The special purpose hardware can then be interfaced with a general purpose computer, which takes care of all the other numerical operations required to solve the equations of motions. It can be seen, from the snapshot-profiles, the virulent notice of gravitational field, gravitational acceleration etc,…In this paper, it has been attempted to carry out simple N-body simulation using simple Mathematica built in codes such as List Plot, List Animate, Bessel J and other simple coding commands.

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