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GRAVITATIONAL COLLAPSE IN SELF-SIMILAR SPACETIMES

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Abstract

Attempts are made to give an alternative description of self-similar spacetimes which is proving to be very substantial and useful in astrophysics and general relativity. The metric for collapsing dust cloud is utilized in this formalism. The nature of gravitational collapse in self-similar spacetimes has been studied in detail and relevant physical interpretations of the results obtained are given. Some interesting results of the calculation have been visualized.

Keywords: self-similarity, gravitational collapse, collapsing dust, homothetic killing vector.

Introduction

Gravitational collapse is an important issue in general relativity and it is widely believed that it may be responsible for high energy objects in our universe. Energy theorems in relativity have shown that under reasonable energy conditions a matter cloud with sufficient mass would undergo a gravitational collapse. General relativistic field equations involve a system of highly nonlinear partial differential equations and hence analyzing a gravitational collapse scenario in general even in spherically symmetric spacetime is virtually impossible. Self-similar spacetimes have therefore been given considerable attention in recent applications. Due to the symmetry property of self-similarity equations in self-similar spacetime become an ordinary differential equation and therefore the study of a phenomena becomes much easier to analyze. In this study we therefore use self-similar spherically symmetric spacetimes to examine the gravitational collapse and its features. In astrophysics and cosmology the self-similar models are of great interest to the relativists and cosmologist alike. It is worthy to start with the very definition of self-similar spacetimes. A self-similar spacetimes is characterized by the existence of a homothetic killing vector field (Joshi, 1993).

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Self-Similar Spacetimes and Path of Photon

A spherical symmetric spacetimes is self-similar if it admits a radial area coordinate ***r*** and an orthogonal time coordinate ***t*** such that for the metric components $\boldsymbol{g_{tt}}$and$\boldsymbol{g_{rr}}$we have

$$g_{tt}(\,kt, kr\,) \;=\; g_{tt}(\,t, r\,)$$

$$g_{rr}(\,kt, kr\,) \;=\; g_{rr}(\,t, r\,)$$

for all $\boldsymbol{k > 0}$. Thus, along the integral curves of the killing vector field all points are similar.

A spherical symmetric spacetimes (SSS) in co-moving coordinates is given by general form

$$ds^2 = -A(t, r)dt^2 + B(t,\; r)dr^2 + r^2C(t,\; r)d\Omega^2$$

where $\boldsymbol{d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2}$. If SSS is self-similar, self-similarity condition must hold, it must have a homothetic killing vectors, which means $\boldsymbol{t\rightarrow kt, r\rightarrow kr}$ and the metric becomes,

$$ds^2 = -A(kt,kr)dt^2 + B(kt,kr)dr^2 + r^2C(kt,kr)d\Omega^2$$

and parameters A,B,C are such that

$$A(t,\; r) = A(kt,kr)$$

$$B(t,\; r) = B(kt, kr)$$

$$C(t,\; r) = C(kt, kr)$$

The collapsing dust cloud is described by the self-similar metric,

$$ds^2 = -dT^2 + R'^2dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (1)$$

where

$$R^{\frac{3}{2}} = \frac{3}{2}\sqrt{F}(t_0\,(r) - t)$$

$$R^{\frac{3}{2}} = \frac{3}{2}\sqrt{F}\left(a - \frac{t}{r}\right)r^{\frac{3}{2}}$$

$$R = (\frac{3}{2})^{\frac{2}{3}} F\left(a - \frac{t}{r}\right)^{\frac{2}{3}} r$$

To check the metric, one can proceed as follows:

$$A(t, r) = -1$$

$$A(kt, kr) = -1$$

$$A(t, r) = A(kt, kr)$$

$$B(t,r) = R'^2 = (\frac{3}{2})^{\frac{2}{3}} F\left(a - \frac{t}{r}\right)^{-\frac{1}{3}}\left(\left(\frac{2t}{3r}\right) + \left(a - \frac{t}{r}\right)\right)$$

$$B(kt,kr) = \left(\frac{3}{2}\right)^{\frac{2}{3}} F\left(a - \frac{kt}{kr}\right)^{-\frac{1}{3}}\left(\left(\frac{2kt}{3kr}\right) + \left(a - \frac{kt}{kr}\right)\right) = R'^2$$

$$B\,(t,r) = B(kt,kr)$$

$$C(t,r) = \frac{R^2}{r^2} = (\frac{3}{2})^{\frac{4}{3}} F^2\left(a - \frac{t}{r}\right)^{\frac{4}{3}}$$

$$C(kt,kr) = (\frac{3}{2})^{\frac{4}{3}} F^2\left(a - \frac{kt}{kr}\right)^{\frac{4}{3}} = \frac{R^2}{r^2}$$

$$C\,(t,r) = C(kt,kr)$$

The above shows that the given metric is self-similar. The radial null geodesicsin this metric is defined by $\boldsymbol{ds^2 = 0}$ and $\boldsymbol{k^{\theta} = k^{\phi} = 0}$(Tolman, 1934). The geodesic equations for $\boldsymbol{k^T}$ and $\boldsymbol{k^r}$ from Lagrangian equation are

$$L = -k^{T^2} + R'^2 k^{r^2} + R^2 k^{\theta^2} + R^2 sin^2\,\theta\, k^{\phi^2} \tag{2}$$

$$\frac{\partial L}{\partial T} = \frac{d}{d\lambda}\left[\frac{\partial L}{\partial k^T}\right]$$

$$k^{r^2}\frac{\partial R'^2}{\partial T} = \frac{d}{d\lambda}\left[\frac{\partial\left(-k^{T^2}\right)}{\partial k^T}\right]$$

$$\frac{dk^T}{d\lambda} + R'\dot{R}'k^{r^2} = 0 \tag{3}$$

Similarly

$$\frac{\partial L}{\partial r} = \frac{d}{d\lambda}\left[\frac{\partial L}{\partial k^r}\right]$$

$$k^{r^2}\frac{\partial R'^2}{\partial r} = \frac{d}{d\lambda}\left[\frac{\partial\left(R'^2 k^{r^2}\right)}{\partial k^r}\right]$$

$$2R'R''k^{r^2} = 4R'\frac{dR'}{d\lambda}k^r + 2R'^2\frac{dk^r}{d\lambda}$$

$$\frac{dk^r}{d\lambda} - \frac{R''}{R'}k^{r^2} + \frac{2}{R'}k^r + \left(\dot{R}'k^T + R''k^r\right) = 0$$

$$\frac{dk^r}{d\lambda} - \frac{\dot{R}'}{R'}k^T k^r + \frac{R''}{R'}k^{r^2} = 0 \qquad (4)$$

Let k^a be tangent to radial null geodesics (i.e, $k^a k_a = 0 = k^a_{;b} k^b$) for the metric in equation(1) and $g_{ab}k^a k^b = 0,$ for null condition. We have for radial null geodesics from equations (3) and (4),

$$g_{TT}k^{T^2} + g_{rr}k^{r^2} = 0$$

$$g_{TT}k^{T^2} = -g_{rr}k^{r^2}$$

$$\frac{k^{T^2}}{k^{r^2}} = \frac{-g_{rr}}{g_{TT}}$$

$$\frac{k^T}{k^r} = \sqrt{\frac{-g_{rr}}{g_{TT}}} = R'$$

$$k^r = \frac{1}{R'}k^T$$

$$\frac{k^T}{k^r} = R' \rightarrow \frac{dT}{dr} = R'$$

$$\frac{dk^T}{d\lambda} = -R'\dot{R}'k^{r'}$$

Where λ is affine parameter. The Kretschmann scalar of the metric is obtained as

$$K = \frac{16(21a^2r^2 - 10art + 5t^2)}{27(-3ar + t)^2(-ar + t)^4}$$

If we assume that ***r* = *r*** and ***t* = *ar***, we get

$$K = \frac{16(21a^2r^2 - 10a^2r^2 + 5a^2r^2)}{27(-3ar + ar)^2(-ar + ar)^4}$$

Which gives us, $K = \infty$

Therefore, the points of unboundedness i.e., $\boldsymbol{K = \infty}$occur at ***(ar, r)***. Self-similarly implies that all variables of physical interest may be expressed in terms of the similarity parameter$\boldsymbol{X = \frac{t}{r}}$**.**

$$X = \frac{t}{r}, \quad t = Xr$$

$$dt = Xdr + rdX$$

$$\frac{dt}{dr} = R'$$

$$\frac{Xdr}{dr} + \frac{rdX}{dr} = R'$$

$$X + \frac{rdX}{dr} = R'$$

$$\frac{rdX}{dr} = R' - X$$

$$\frac{dX}{dr} = \frac{R' - X}{r}$$

$$\int \frac{1}{r}\, dr = \int \frac{1}{R' - X}\, dX$$

$$ln\, r = -ln[-R' + X]$$

The above is the possible path of photon in gravitationally collapsing objects.

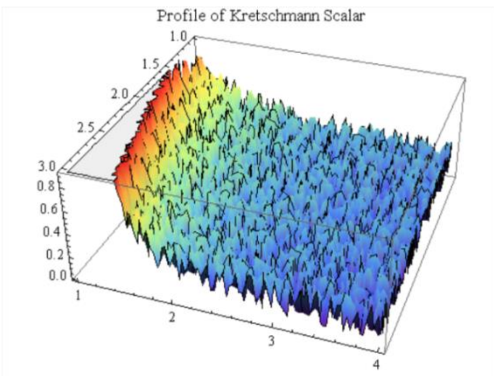

Figure: The profile of the Kretschmann scalar of the metric with an orthogonal time coordinate t and a radial area coordinate r

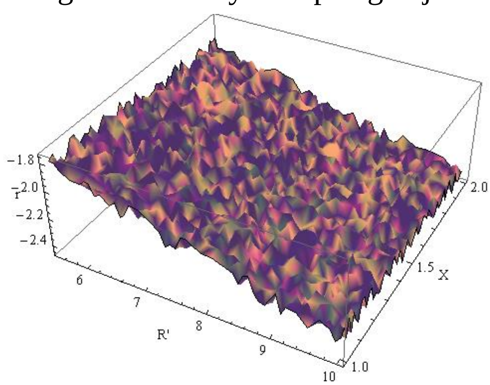

Figure: The profile of the path of photon in gravitationally collapsing objects

Gravitational Collapse in Self-Similar Spacetimes

We would like to examine the determination of curvature strength of the naked singularity in order to decide on its seriousness and physical relevance and the mathematical calculations of Christoffel Symbols, Riemann Tensors, Ricci Tensors and Kretschmann Scalar of the metric.

The collapsing dust cloud is described by the self-similar metric,

$$ds^2 = -dT^2 + R'^2 dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (5)$$

where
$$R^{\frac{3}{2}} = \frac{3}{2}\sqrt{F}(t_0(r) - t)$$

$$R^{\frac{3}{2}} = \frac{3}{2}\sqrt{F}\left(a - \frac{t}{r}\right) r^{\frac{3}{2}}$$

$$R = (\frac{3}{2})^{\frac{2}{3}} F \left(a - \frac{t}{r}\right)^{\frac{2}{3}} r$$

Let $\boldsymbol{b} = \left(\frac{3}{2}\right)^{\frac{2}{3}} \boldsymbol{F}$ and which gives $R = b\left(a - \frac{t}{r}\right)^{\frac{2}{3}} r$

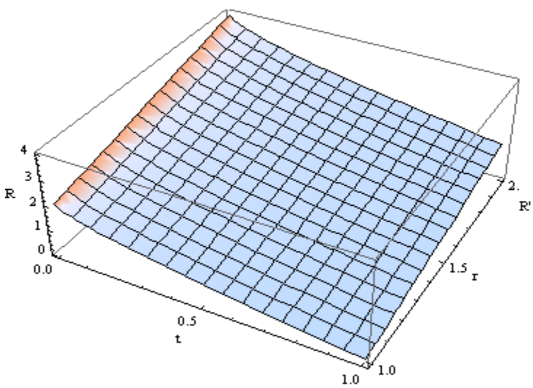

Figure: The profile of the homothetic coordinate R(t, r) in self-similar spacetimes.

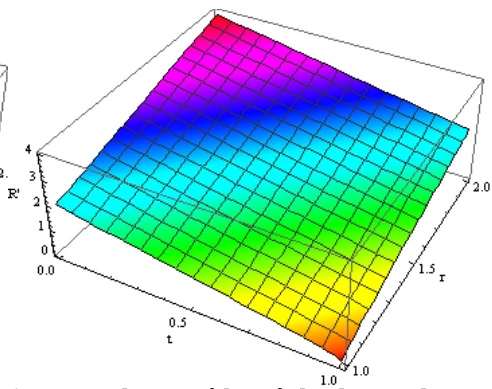

Figure: The profile of the homothetic coordinate R'(t, r) in self-similar spacetimes.

Here, ***b*** is constant and after differentiation we get

$$R' = \frac{b\,(3ar - t)}{3r\left(a - \frac{t}{r}\right)^{\frac{2}{3}}} \tag{6}$$

From metric, $g_{11} = -1,\ g_{22} = R'^2$

$$g_{33} = R^2, \qquad g_{44} = R^2 sin^2\Theta$$

To find some calculations, we have to use the following equations,

Christoffel Symbol are denoted by, $\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda k}\left(g_{k\nu,\mu} + g_{\mu k,\nu} - g_{\mu\nu,k}\right)$

Riemann Tensor $R_{abcd} = g_{d\in} R^{\in}_{abc}$

Kritchmann Scalar $K = R^{abcd} R_{abcd}$

Ricci Tensor $R_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma} - \Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma}$

Gravitatinal collapse of Bose-Einstein condensate dark matter halos

As a first step in the study of the time dynamics of the gravitationally bounded Bose-Einstein condensates, we have to chose a variational tril wave function. Instead of fixing it in an arbitrary way (by assuming, for example, that the initial density profile of the condensate has a Gaussian form), we require that $|\psi|^2$ satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\vec{\upsilon}) = 0 \tag{7}$$

For the density of the Bose-Einstein condensate we assume a general form

$$\rho(r,t) = \rho_0(t) + \rho_1(t)\rho_2(r), \tag{8}$$

where $\rho_0(t), \rho_1(t)$ and $\rho_2(r)$ are arbitrary functions of ***t*** and ***r*** to be determine.From a physical point of view, the trail density profile $\rho(t)$ is the sum of two terms, the first representing a "cosmological" type homogeneous term $\rho_{hom}(t)$, while the second term $\rho_{\text{inhom}}(t, r)$represent the effect of the time-independent inhomogeneities in the dark matter halo. The inhomogeneous term is assumed to be separable in the variables t and r, so that $\rho_{inhmo}(t, r) = \rho_1(t)\rho_2(r)$.

Substitution in equations $\vec{v} = H(t)\vec{r}$ and $\rho(r,t) = \rho_0(t) + \rho_1(t)\rho_2(r)$, into the continuity equation Eq.(7) gives

$$\dot{\rho}_0(t) + 3\rho_0(t)\frac{\dot{R}(t)}{R(t)} + \rho_2(r) \times \left[\dot{\rho}_1(t) + 3\rho_1(t)\frac{\dot{R}(t)}{R(t)} + \frac{\dot{R}(t)}{R(t)}\rho_1(t) r \frac{\acute{\rho}_2(r)}{\rho_2(r)}\right] = 0 \qquad (9)$$

We determine the function $\rho_2(r)$ by imposing the condition rρ'_2 (r)/ρ_2 (r) = constant = α > 0, which leads first to

$$\rho_2(r) = C_1 r^{\alpha}, \qquad (10)$$

where C_1 is an arbitrary constant of integration. Next we require that the term in the square bracket of Eq.(9) vanishes. Therefore, Eq.(9) gives the following two independent differential equations for the determination of the functions ρ_0(t) and ρ_1(t),

$$\dot{\rho}_0(t) + 3\rho_0(t)\frac{\dot{R}(t)}{R(t)} = o, \qquad (11)$$

And

$$\dot{\rho}_1(t) + (3 + \alpha)\rho_1(t)\frac{\dot{R}(t)}{R(t)} = 0, \qquad (12)$$

respectively. Hence, the general solution of Eq.(9) can be obtain as

$$\rho(r,t) = \frac{1}{R^3(t)}\left[a_0 + b_0\frac{r^{\alpha}}{R^{\alpha}(t)}\right], \qquad (13)$$

where a_0 and b_0 are arbitrary constants of integration. Since at the vacuum boundary of the condensate, where r = R(t), the density must satisfy the condition ρ [R(t), t] = 0, At ≥ 0, we obtain for the two integration constants the condition $a_0 + b_0 = 0$.

Concluding Remarks

In this thesis some fundamental notions and basics of self-similar spacetime are given and attempts are made to derive the path of photon using simple tractable methods. The Kretschmann scalar is also calculated for spherical symmetric spacetimes metric. Self-similar solutions are derived and visualizations of some results are done with the help of mathematica. Autonomous phase plane, self-similar and scaling exact solutions with scalar field are given and some animation plots are visualized using mathematica.

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Appendix

It is necessary to compute the Christoffel symbols for equation (1), from which we can get the curvature tensor. If we use labels (1,2,3,4) for (T, r, θ, ϕ) in the usual way, non zero Christoffel symbols are given by using Tensorpak. m package,

$$\Gamma^2_{12} = \frac{2t}{3(3a^2r^2 - 4art + t^2)}\ , \Gamma^3_{13} = \frac{2}{-3ar + 3t}\ , \Gamma^4_{14} = \frac{2}{3ar + 3t}\ ,$$

$$\Gamma^2_{21} = \frac{2t}{3(3a^2r^2 - 4art + t^2)}\ , \Gamma^1_{22} = \frac{2b^2(3ar - t)t}{27r^2(ar - t) - (a - \frac{t^{\frac{2}{3}}}{r})}\ , \Gamma^2_{22}$$

$$= \frac{2t^2}{-9a^2r^3 + 12ar^2 - 3rt^2}\ ,$$

$$\Gamma^3_{23} = \frac{3ar - t}{3ar^2 - 3rt}\ , \Gamma^4_{24} = \frac{3ar - t}{3ar^2 - 3rt}\ , \Gamma^3_{31} = \frac{2}{-3ar + 3t}\ , \Gamma^3_{32}$$

$$= \frac{3ar - t}{3ar^2 - 3rt}\ ,$$

$$\Gamma^1_{33} = -\frac{2}{3}b^2r\left(a - \frac{t}{r}\right)^{\frac{1}{3}}, \Gamma^2_{33} = \frac{3a(-ar + t)}{3ar - t}\ , \Gamma^4_{34} = \cot\theta\ , \Gamma^4_{41}$$

$$= \frac{2}{-3ar^2 + 3t}\ ,$$

$$\Gamma^4_{42} = \frac{3ar - t}{3ar^2 - 3rt}\ , \Gamma^4_{43} = \cot\theta\ , \Gamma^1_{44} = -\frac{2}{3}b^2\, r\left(a - \frac{t}{r}\right)^{\frac{1}{3}} \sin^2\theta\ , \Gamma^2_{44}$$

$$= -\frac{3r(ar - t)\sin^2\theta}{3ar - t}\ ,$$

$$\Gamma^3_{44} = -\cos\theta\sin\theta\ ,$$

From these we get the following nonvanishing components of the Riemanntensor

$$R_{1212} = \frac{2(3ar + t)}{9(3ar - t)(-ar + t)^2}\ , R_{1221} = -\frac{2(3ar + t)}{9(3ar - t)(-ar + t)^2}\ , R_{1313}$$

$$= -\frac{2}{9(-ar + t)^2}\ ,$$

$R_{1331} = \frac{2}{9(-ar+t)^2}$, $R_{1414} = -\frac{2}{9(-ar+t)^2}$, $R_{1441} = \frac{2}{9(-ar+t)^2}$, $R_{2112} = -\frac{2(3ar+t)}{9(3ar-t)(-ar+t)^2}$,

$R_{2121} = \frac{2(3ar+t)}{9(3ar-t)(-ar+t)^2}$, $R_{2323} = \frac{4t}{9(-3ar-t)(-ar+t)^2}$, $R_{2332} = -\frac{4t}{9(-3ar-t)(-ar+t)^2}$,

$R_{2424} = \frac{4t}{9(-3ar+t)(-ar+t)^2}$, $R_{2442} = -\frac{4t}{9(-3ar+t)(-ar+t)^2}$, $R_{3113} = \frac{2}{(3ar-3t)^2}$,

$R_{3131} = -\frac{2}{(-3ar-3t)^2}$, $R_{3223} = -\frac{4t}{9(-3ar+t)(-ar+t)^2}$, $R_{3232} = \frac{4t}{9(-3ar+t)(-ar+t)^2}$,

$R_{3434} = \frac{4}{9(-ar+t)^2}$, $R_{3443} = -\frac{4}{9(-ar+t)^2}$, $R_{4114} = \frac{2}{(3ar-3t)^2}$, $R_{4141} = -\frac{2}{(3ar-3t)^2}$,

$R_{4224} = -\frac{4t}{9(-3ar+t)(-ar+t)^2}$, $R_{4242} = \frac{4t}{9(-3ar+t)(-ar+t)^2}$, $R_{4334} = -\frac{4}{9(-ar+t)^2}$,

$R_{4343} = \frac{4}{9(-ar+t)^2}$,

We get non-zero values of Ricci Tensor

$$R_{11} = -\frac{2}{3(3\,a^2r^2 - 4art + t^2)} , R_{22} = \frac{2b^2(-3ar+t)}{27r^2(\,ar-t)\left(a-\frac{t^{\frac{2}{3}}}{r}\right)} , R_{33} = -\frac{2b^2r(a-\frac{t}{r})^{\frac{1}{3}}}{9ar-3t} , R_{44} = -\frac{2b^2r(a-\frac{t}{r})^{\frac{1}{3}}sin^2\theta}{9ar-3t} ,$$

Scalar Curvature is, $R = -\frac{4}{3(3a^2r^2-4art+t^2)}$,

and Krichman Scalar is, $K = \frac{16(21a^2r^2-10\,art+5t^2)}{27(-3ar+t)^2(ar+t)^2}$.