SOLVING QUANTUM MECHANICAL HARMONIC OSCILLATOR AND CHAOTIC QUANTUM LINEAR MAP

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Abstract

Fundamental motion of quantum mechanical harmonic oscillator and chaotic quantum linear map has been investigated using numerical methods and visualization of the results. Periodic chaotic nature is found in visualization of quantum linear map.

Keywords: Harmonic Oscillator and Chaotic Quantum Linear Map

Introduction

Numerical solutions of the time-independent Schrodinger equation have been discussed in various conditions. These solutions utilize numerical techniques for solving a differential time evolution for the time-dependent Schrodinger equation. Many quantum mechanical research problems that are answerable to solution (for example, the behavior of electrons on a small lattice) by using matrix mechanics. In the quantum linear map, although the square of commutator can increase exponentially with time, a simple operator does not scramble but performs chaotic motion in the operator basis space determined by the classical linear map.

The Formalism

The harmonic oscillator is also answerable to analytical solution. Unlike the infinite square well the solutions are unfamiliar and concerned in the initial state.

The infinite square wellis defined as

$$V_{\text{inf}}(x) = \begin{cases} 0 & \text{if } 0 < x < a, \\ \infty & \text{otherwise.} \end{cases} \tag{1}$$

The Hamiltonian is given by

$$H_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_{\text{inf}}(x), \tag{2}$$

where m is the mass of the particle.

The eigenstates are well known:

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) & \text{if } 0 < x < a, \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

with eigenvalues,

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$$E_n^{(0)} = \frac{n^2\pi^2\hbar^2}{2ma^2} \equiv n^2 E_1^{(0)}. \tag{4}$$

The quantum number $n = 1,2,3,...$ takes on a positive integer value.

The Harmonic Oscillator

For the infinite square well, the harmonic oscillator potential is $V_{HO} = \frac{m\omega^2 x^2}{2}$. In the following all units of distance will be in terms of the square well width a, and all units of energy will be in terms of the (unperturbed) ground state, $E_1^{(0)}$. We will use lower case letters to denote dimensionless energies. The potential V_{HO} can be written in terms of the infinite square well length and energy scales as

$$v_{HO} = \frac{V_{HO}}{E_1^{(0)}} = \frac{\pi^2}{4}\left(\frac{\hbar\omega}{E_1^{(0)}}\right)^2\left(\frac{x}{a} - \frac{1}{2}\right)^2, \tag{5}$$

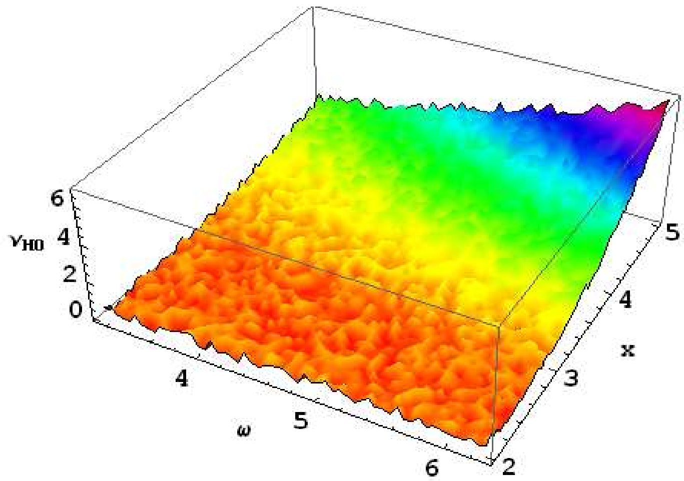

Figure 1 3D Profile of v_{HO} in terms of ω and x.

so that the dimensionless parameter $\frac{\hbar\omega}{E_1^{(0)}}$ determines the stiffness of the harmonic oscillator potential. We expect that for low energy states (the ground state), the solution should be identical to that of the harmonic potential alone, because the wave function will be sufficiently restricted to the central region of the harmonic oscillator potential so that it will not feel the walls of the infinite square well. High energy states will not be well described by the harmonic oscillator results, because they will be primarily governed by the infinite square well. We first use

$$H_{nm} = \delta_{nm}E_n^{(0)} + \frac{2}{a}\int_0^a dx \sin\left(\frac{n\,\pi\,x}{a}\right)V(x)\sin\left(\frac{m\,\pi\,x}{a}\right) \tag{6}$$

with the potential given by equation (5). The result is

$$= \delta_{nm}\left\{n^2 + \frac{\pi^2}{48}\left(\frac{\hbar\omega}{E_1^{(0)}}\right)^2\left(1 - \frac{6}{(\pi\,\mathrm{n})^2}\right)\right\} + (1-\delta_{nm})\left(\frac{\hbar\omega}{E_1^{(0)}}\right)^2 g_{nm} \tag{7}$$

where. $$g_{mn} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (8)$$

The g_{nm} remain of order unity close to the diagonal, but for large n the diagonal elements grow as n^2, so the off-diagonal elements become negligible in comparison.

Quantum Linear Map

The operator scrambling in the quantum linear map is an instructive quantum mechanical model with many properties exactly solvable. Before we study this model in detail, we first briefly review the classical linear map.

The classical linear map is the linear automorphism of the unit torus phase space given by

$$\begin{pmatrix} q \\ p \end{pmatrix} \rightarrow \begin{pmatrix} a & \mathrm{b} \\ c & \mathrm{d} \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \bmod 1 \quad (9)$$

where the matrix $\mathrm{M} = \begin{pmatrix} a & \mathrm{b} \\ c & \mathrm{d} \end{pmatrix} \in SL(2,Z)$. The determinant is equal to one so that this map isarea preserving (canonical). Also it preserves the periodic boundary condition of the torus as M has integer valued entries. The Lyapunov exponents $\lambda_{\pm}$ of the linear map are given by the logarithm of the eigenvalues of M. When TrM > 2, this map is hyperbolic and has $\lambda_{+} > 0 (\lambda_{+} + \lambda_{-} = 0)$. The chaotic linear map is known to be fully ergodic and mixing. If consider a simple case with a = 2; b = 1; c = 3; d = 2 and $\lambda_{\pm} = \log(2 \pm \sqrt{3})$.The linear map can be quantized on the square torus with finite Hilbert space. We define $|q_n\rangle$ and $|p_n\rangle$ to be position and momentum eigenstates with n=0, 1, … ,K-1,where K is dimension of the Hilbert space. The position and momentum translation operators are defined through $\hat{\tau}|q_n\rangle = |q_{n+1}\rangle$ and $\hat{\sigma}|p_n\rangle = |p_{n+1}\rangle$. Hence $\hat{\sigma}$ and $\hat{\tau}$ can be represented as Z_K rotor operators,

$$\hat{\sigma} = \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \omega \cdots 0 \\ \vdots & \vdots \ddots \vdots \\ 0 & 0 \cdots \omega^{K\text{-}1} \end{pmatrix}, \hat{\tau} = \begin{pmatrix} 0 \cdots 0 & 1 \\ 1 \cdots 0 & 0 \\ \vdots \ddots \vdots & \vdots \\ 0 \cdots 1 & 0 \end{pmatrix} \quad (10)$$

where $\omega = e^{-2\pi i/K}$. $\hat{\sigma}$ and $\hat{\tau}$ satisfy $\hat{\sigma}^K = \hat{\tau}^K = 1$with the commutation relation given by $\sigma\tau = \omega\tau\sigma$.

In the position representation, the quantum propagator for the quantum linear map with

$$M = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

is obtained by path integral method and takes this form

$$\hat{U}(q',q) = \left(\frac{i}{K}\right)^{1/2} \exp\left[\frac{i\pi}{K}(2q^2 - 2qq' + 2(q')^2\right] \quad (11)$$

where $q, q' = 0,1,..., K-1$ label the position eigenstates. For any classical observable, one can associate a quantum observable operator $\hat{O}(f)$, which respects

$$\hat{U}^{\dagger}\hat{O}(f)\hat{U} = \hat{O}(f \circ M)$$

This equation usually holds in the limit $N \to \infty$. However due to the map being linear here, it holds even at finite N. Therefore, for $\hat{\sigma}$ and $\hat{\tau}$,

$$\hat{U}^{\dagger}\hat{\sigma}\hat{U} \sim \hat{\sigma}^2\hat{\tau}\,\hat{U}^{\dagger}\hat{\tau}\hat{U} \sim \hat{\sigma}^3\hat{\tau}^2. \quad (12)$$

This result indicates that for any operator of the form $\hat{O} = \hat{\sigma}^q\hat{\tau}^p$ under unitary time evolution, it performs chaotic motion in the operator basis space spanned by $\hat{B}_{mn} = \sigma^m\tau^n$ (with $m,n = 0,1,..., K-1$) which satisfies $Tr\hat{B}^{\dagger}{}_{mn}\hat{B}_{m'n'} = K\delta_{m,m'}\delta_{n,n'}$. The evolution of (q,p) is determined by the classical linear map defined in equation (9) and gives rise to the exponential growth of the square of commutator $C(t) = \left\langle\left[\hat{O}(t),\hat{O}\right]\left[\hat{O}(t),\hat{O}^{\dagger}\right]\right\rangle$,i.e., C(t) ~ $e^{2\lambda+t}$ when t is smaller than the Lyapunov time $t_L = \log K/\lambda_+$ .When $t > t_E$, the quantum correction becomes important and $C(t)$ stops to increase exponentially.

The exponential growth of $C(t)$ has a classical origin and is not related withthe operator scrambling. Under unitary time evolution, $\hat{O}(t)$ is always a basis operator anddoes not become more complicated.

To realize operator scrambling, we consider the quantum linear map perturbed by anonlinear shear. The new composite Floquet operator is,

$$\hat{U} = \hat{U}_1\hat{U}_2 \quad (13)$$

where $\hat{U}_2$ is the quantum linear map defined in equation(11) and $\hat{U}_1$ describes a nonlinear shear

$$\langle q'|\hat{U}_1|q\rangle = \exp\left[i\frac{kK}{2\pi}\left(\sin\left(\frac{2\pi q}{K}\right) - \frac{1}{2}\sin\left(\frac{4\pi q}{K}\right)\right)\right]\delta_{q,q'} \quad (14)$$

which will not have much influence on the early time dynamics as long as k is small.

Conclusion

In this paper we describe the Hamiltonian equation, harmonic oscillator and the quantum linear map. In the quantum linear map although the square of the commutator $C(t)$ can grow exponentially with the time and the quantum operator does not scramble at all. The operator scrambling can occur once if one makes some modification in the Floquetoperator. So that the quantum chaos is not always associated with the exponential growth of the square of the commutator $C(t)$. The non-linear shear equation simply shows that only at some selection point of the dynamical variable, it gives chaotic nature. Visualization gives this nature clearly.

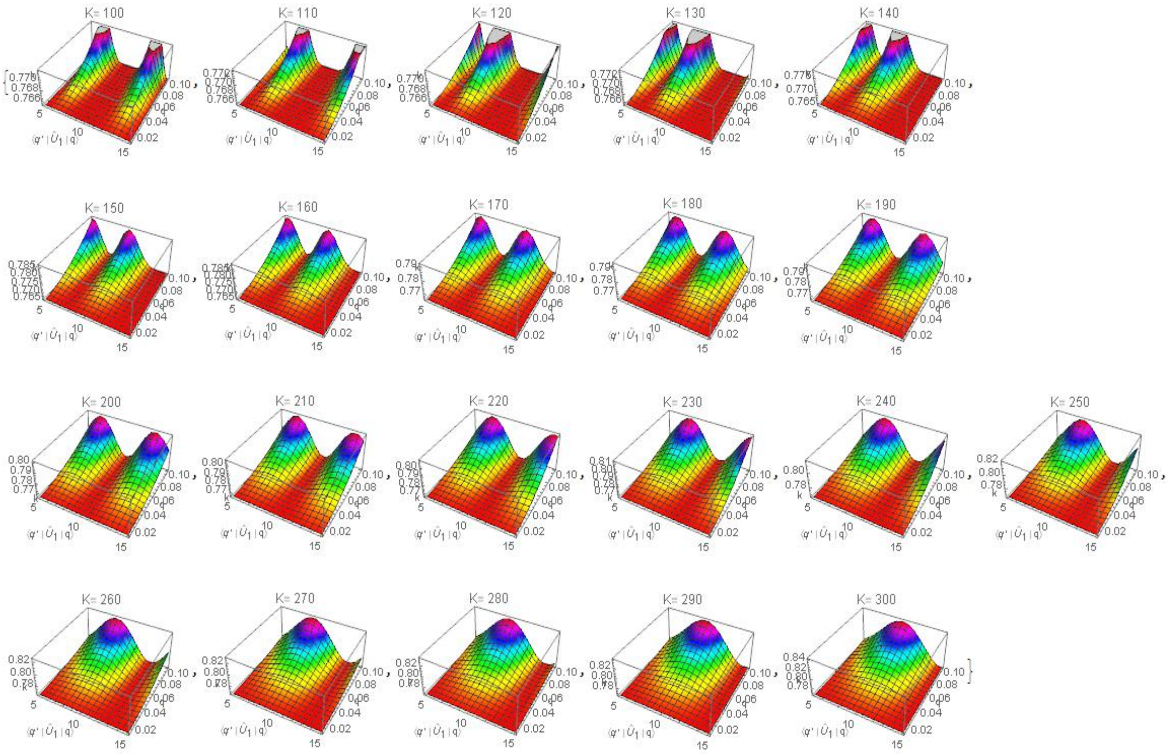

Figure 2 3D Profile of $\langle q'|\hat{U}_1|q\rangle$ in terms of k and q (Real).

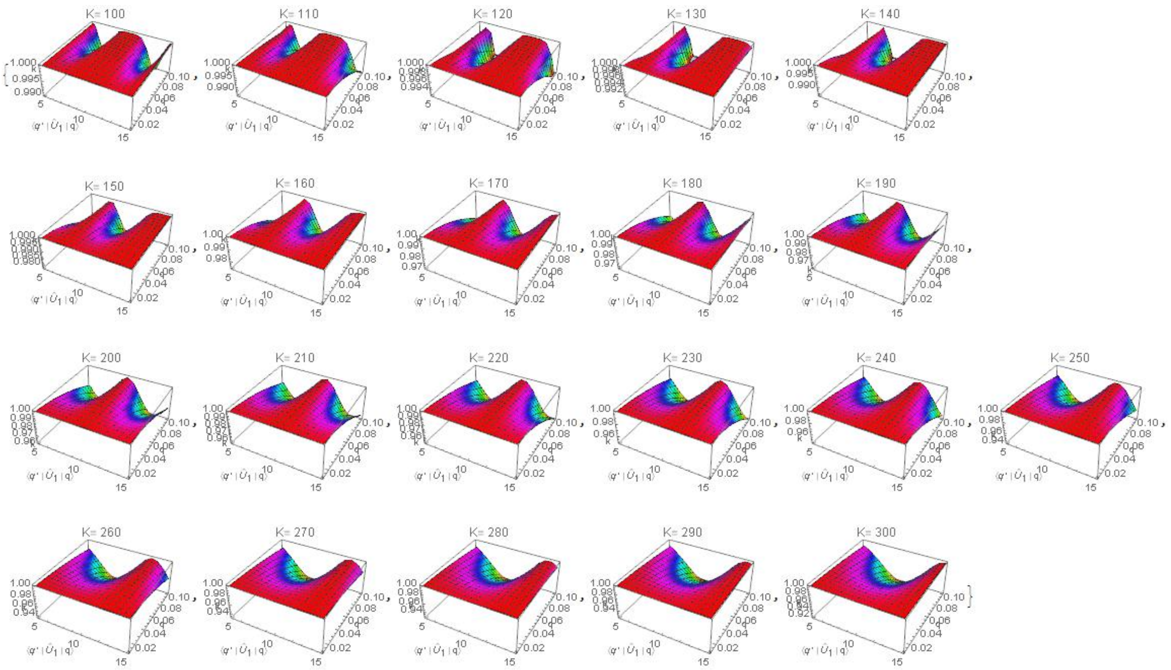

Figure 3 3D Profile of $\langle q'|\hat{U}_1|q\rangle$ in terms of k and q (Imagiary).

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