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WAVELETS ON GROUPS

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Abstract

This paper is an expository survey of basic concepts of Wavelet Analysis. We shall discuss wavelets in L^2 (R) and L^2 (G), where G is a locally compact abelian Group, in particular a Lie group. A discussion of basic facts of Topological groups, Differentiable Manifolds and Lie groups are also included.

Keywords: Haar measure, Topological group, Differentiable Manifold, Lie Group, Wavelets.

1. Introduction

The classical Fourier theory is concerned with the study of the Fourier transform $\hat{f}$ of a given function f:

(1) $$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx$$

and its inversion problem i.e. studying conditions under which the following inversion relation holds:

(2) $$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi x}dx.$$

The corresponding Foureir sereis theory is the investigation of the validity of the relation.

(3) $$f(x) = \sum_{n\in Z} \hat{f}(n)e^{inx}$$

in various modes of convergence, where

(4) $$\hat{f}(n) = \int_{-\infty}^{\infty} f(x)e^{-inx}dx.$$

The infinite series

(5) $\sum_{n\in Z} \hat{f}(x)e^{inx}$ is called the Fourier series of f and the numbers $\hat{f}(x)$ are called the fourier coefficients of f. The Euler relation, $e^{ix} = \cos x + i\sin x$,

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shows that the Fourier sereis in (5) is infact a series in sine and cosine. In application, we usually have to approximate a function by its Fourier series,

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(x)e^{inx}$$

and thus have to compute the infinite integral

$$\hat{f}(n) = \int_{-\infty}^{\infty} f(x)e^{-inx}dx.$$

Except for very nice functions, this integral cannot be evaluated in closed form.

Again we have to approximate this infinite integral on some suitable finite interval. It is therefore desirable that the integrand decays at infinity.

The problem here is that sine and cosine functions do not decay at infinity i.e., $|\sin x|, |\cos x| \not\longrightarrow 0$ as $|x| \to \infty$. They remain oscillatory on the whole real line.

Wavelets are an attempt to replace sine and cosine functions with function having sufficient rate of decay at infinity.

Wavelets (on *P*) were introduced in early nineteen eighties by Morlet, Arens, Fourgeau and Giard. Later the mathematical foundations of the wavelet theory was laid down by I. Daubechies and Y. Meyers [7]. This paper is an exposition of the extension of wavelet theory from *R* to topological groups.

2. Fourier Analysis on Groups

In this section we briefly discuss Fourier Analysis on Groups. For details, we refer to [8].

2.1 Haar Measure

2.1.1Definition. A **topological group** *G* is a group which is also a topological space such that the group operations

$$G \times G \quad \to \quad G$$

$$(x, y) \mapsto xy,$$

and

$$G \to G$$
$$x \mapsto x^{-1}$$

are continuous.

A topological group G is locally compact if it is locally compact as a topological space.

2.1.2 Proposition. Let G be a locally compact abelian group (LCA). Then there exists a non-negative regular and translation invariant Borel measure on G. This measure is called the ***Haar measure*** on G. [8].

Some function spaces of interest on G are as follows:

- $C(G)$ denotes the set of all continuous complex functions on G.
- $C_C(G)$ denote the set of all continuous complex functions on G with compact support.
- $L^P(G), 1 \le p < \infty$ the set of all Borelmeasurable functions on G such that

$$\left(\int_G | f |^P \, dx \right)^{1/p} < \infty$$

where dx is the Haar measure on G.

2.2 The Dual Group and the Pontryagin Duality

2.2.1 Definition. Let G be a LCA group. A complex function γ on G is called a ***character*** of G if for all $x, y \in G$,

(i) $|\gamma(x)| = 1$

(ii) $\gamma(xy) = \gamma(x)\gamma(y)$.

2.2.2 Remark.

(1) γ is a homomorphism of the group G and the multiplicative group S^1 of the unit circle in X.

(2) The example of a character is the exponential map

$$R \to S^1$$
$$x \mapsto e^{ix}.$$

2.2.3 Definition. Let Γ be the set of all continuous characters of G. Γ is a group with respect to the addition defined by

$$(\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x), \qquad (\gamma_1, \gamma_2 \in \Gamma, x \in G)$$

$(\Gamma, +)$ is called the ***dual group of G***.

It is customary to write the "duality notation" (x, γ) for $\gamma(x)$.

2.2.4 Theorem (The Pontryagin duality)

Let G be a LCA group and Γ be its dual group. Let $\hat{\Gamma}$ be the dual group of Γ. Then $\hat{\Gamma} \simeq G$. [8]

2.3 The Fourier Transform on Locally Compact Abelian Group

Let G be a LCA group and Γ be its dual group. Let $f \in L^1(G)$. A function $\hat{f}$ defined on Γ by

$$\hat{f}(\gamma) = \int_G f(x)(-x, \gamma)dx \,, (\gamma \in \Gamma)$$

is called the **Fourier transform** of f.

This generalization of the ***classical*** Fourier transform on R to a LCA group G is only too natural, since R is also a LCA group.

As may be expected the following classical results still hold:

(1) $\int_G f(x)g(x)dx = \int_\Gamma \hat{f}(\gamma)\hat{g}(\gamma)d\gamma,$ (Parseval)

(2) $\| f \|_2 = \| \hat{f} \|_2,$ (Plancherel) [8]

2.4 Lie groups

In doing Fourier Analysis especially wavelet Analysis, it is sometimes necessary to consider smooth functions also.

For this purpose we have to consider topological spaces which have also differentiable structure.

A **differentiable manifold** is a Hausdorff topological space X such that each point in X has a neighbourhood homeomorphic to an open set in R^n.

2.4.1 Definition. A topological group G which is also a differentiable manifold such that the maps

$$\begin{array}{ccc} G \times G & \to & G \\ (x, y) & \mapsto & xy \end{array} \quad \text{and}$$

$$\begin{array}{ccc} G & \to & G \\ x & \mapsto & x^{-1} \end{array}$$

are smooth is called a **Lie group**.

For our purpose the following matrix Lie groups will suffice.

Let k be the real field R or the complex field X.

Let $M_n(k)$ be the set of all $n \times n$ matrices with entries from k.

(1) The general linear group $GL_n(k) = \{A \in M_n(k) : \det A \neq 0\}$.

(2) The special linear group $SL_n(k) = \{A \in GL_n(k) : \det A = 1\}$.

(3) The orthogonal group $O_n(k) = \{A \in GL_n(k) : \overline{A^T} A = I_n\}$.

Clearly $GL_n(k), SL_n(k)$ and $O_n(k)$ are groups with respect to ordinary matrix multiplication.

Each $n \times n$ matrix A can be identified as a point of k^{n^2}.

So $GL_n(k)$, $SL_n(k)$ and $O_n(k)$ are also topological groups with the topology induced by k^{n^2}. For details we refer to [1].

3. Wavelets

3.1 Wavelets on *P*

3.1.1 Definition. Let $\psi \in L^2(R)$ with sufficient rate of decay at infinity.

Consider the family

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \ (j,k \in Z)$$

of translations and dialations of ψ .

Suppose that $\{\psi_{j,k}\}$ forms an orthonormal basis for $L^2(R)$; i.e.,

$$\langle \psi_{j,k}, \psi_{l,m} \rangle = \delta_{j,l}\delta_{k,m}$$

$$f = \sum_{j,k \in Z} \langle f, \psi_{j,k} \rangle \psi_{j,k}.$$

Then ψ is called a mother wavelet and the system $\{\psi_{j,k}\}$ is called a wavelet basis.

Example (1). The Haar function is a wavelet with compact support.

$$\psi(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < \frac{1}{2}, \\ -1 & \text{if} \quad \frac{1}{2} \le x < 1, \\ 0 & \text{elsewhere} \end{cases}$$

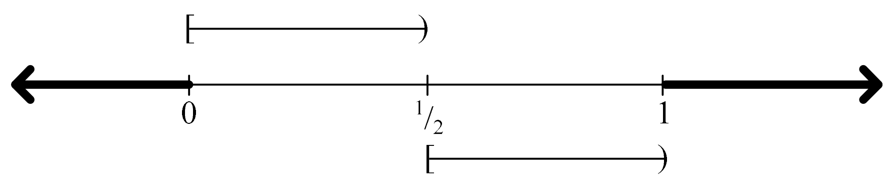

Figure (1)

Example (2). The Gaussian function is a smooth wavelet with fast rate of decay at infinity.

$$\psi(x) = e^{-x^2}$$

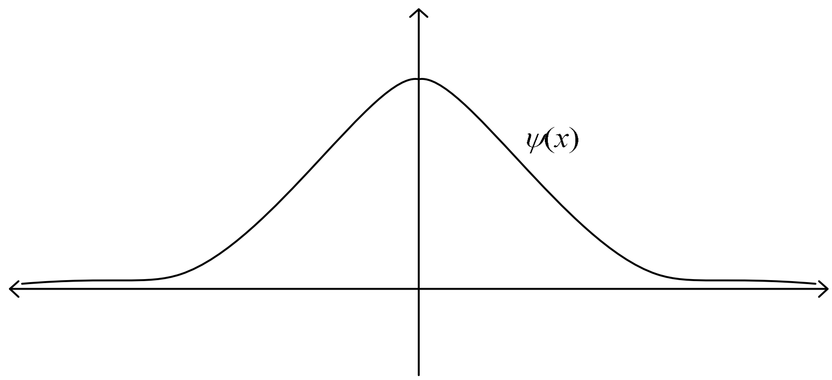

Figure (2)

3.2 Group theoretical approach to wavelets.

We recall some concepts of group representations [9].

3.2.1 Definition. Let G be a locally compact group. A **unitary representation** of G is a pair (π, H) where H is a Hilbert space and π is a continuous homomorphism of G into the group $U(H)$ of unitary operators on H that is the operations are continuous.

$$\pi(xy) = \pi(x)\pi(y),$$

$$\pi(x^{-1}) = (\pi(x))^{-1} = (\pi(x))^{*} \quad \text{for } x, y \in G.$$

3.2.2The *Affine* Group

Let $G = \{(a,b) \in R^{\times} \times R, a \neq 0\}$.

Define the operation $(a,b)\cdot(c,d) = (ac, d + \frac{b}{c})$.

Then G is a group called the "***Affine*** group" with $(a,b)^{-1} = (a^{-1}, -ab)$.

For $g = (a,b) \in G$, define

$$(T_g\psi)(x) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right) = \frac{1}{\sqrt{|a|}}\psi(g^{-1}(x))$$

where $\psi \in L^2(R)$.

Then the mapping,

$$T : G \to U(L^2(R))$$

$g \mapsto T_g$ is a unitary representation of G on $L^2(R)$, [11].

3.2.3 The continuous wavelet transform

Let $\psi \in L^2(R)$ be a wavelet. Then the mapping $W_\psi : L^2(R) \to L^2(R^\times \times R)$ defined by

$$\begin{aligned}(W_\psi f)(g) &= \int_R f(x)\overline{(T_g\psi)(x)}dx \\ &= \langle f, T_g\psi \rangle \\ &= \langle f, \psi_{a,b} \rangle\end{aligned}$$

is called the continuous wavelet transform.

The main purpose of wavelet Analysis is [like that of Fourier Analysis] to look for conditions such that the **Calderon**reproducing formula

$$\int_G (W_\psi f)(g)\psi(x)d\lambda(g) = f$$

holds, where $\lambda$is the Haar measure on G [11].

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