SCATTERING PARAMETERS FOR PROTON-NEUTRON SCATTERING WITH YUKAWA POTENTIAL

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Abstract

The purpose of the research work is to calculate the scattering parameters for two-nucleon system with Yukawa potential. We have calculated the scattering phase shift, effective range and scattering length. To calculate the phase shift, the transition matrix (T-matrix) is formulated from Lippmann-Schwinger equation for two-body scattering state. The Lippmann-Schwinger equation is the Schrödinger equation with boundary condition. The numerical values of T-matrix are obtained by using Gauss elimination Method with FORTRAN code. And then, the effective range and scattering length are calculated by using the phase shift.

Keywords: Scattering phase shift, effective range, scattering length

Introduction

Nucleon-nucleon interactions are studied based on scattering experiments. The interaction between two nucleons is basis for all of nuclear physics. The scattering is related to the behavior of the interaction. The simplest bound system in nature is the deuteron which consists of a neutron and a proton. A phenomenological nucleon-nucleon interaction based on the meson exchanges. The meson exchange idea introduced by Yukawa in 1934 is a good starting point to examine nucleon-nucleon interaction. In the Yukawa picture, the interaction between two nucleons is mediated by the exchange of mesons.

Protons and neutrons are lowest energy bound states of quarks and gluons. When we put two or more of these particles together, they interact, scatter and sometimes form bound states due to the strong interaction. On first tries to extract the nucleon-nucleon interaction from the nucleon-nucleon scattering data. For neutron scattering, there are two major sources for incident beam. At low energies, neutrons from nuclear reactors may be used. At higher energies, one can make use of neutrons produced a beam of protons. However, both the intensity and the energy resolution of neutron beam obtained in these ways are much more limited than those for proton beams. As a result neutron scattering is a more difficult experiment than those with protons. The information obtained from proton-neutron (np) and proton-proton(pp) scattering may not be any different from that in np- and pp-scattering is whether the neutron or the proton is the target. If nuclear force is charge independent, the results of pp- and nn-scattering can only be different by the contribution made by coulomb interaction. A comparison of pp- and nn-scattering results is the charge independence of nuclear interaction.

Phase shift, Scattering length and effective range are scattering parameters. These parameters provide a useful way to parameterize information on low-energy nucleon-nucleon scattering. These parameters may be related to observations other than NN-scattering, such as deuteron binding energy. In addition, very accurate results can be obtained for the np- system by scattering slow neutrons off protons in hydrogen atoms. For these reasons, a great deal of attention is devoted to the measurement and understanding of these parameters. Any potential

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shape can be used to obtain these parameters by a proper choice of the range and depth of the well. We can relate these parameters to the properties of ground state of the deuteron.

In our calculation, the Yukawa type Malfliet-Tjon potential is used. The Malfliet-Tjon potential is the Yukawa type potential including also 'soft core' repulsion. This potential is a central potential independent of spin and isospin.

The nucleon-nucleon interaction in the ${}^1S_0$ channel is a superposition of two Yukawa potential one of which is repulsive, i.e.

$$V(r) = V_A \frac{e^{-\mu_A r}}{r} + V_R \frac{e^{-\mu_R r}}{r} \; with \;\; \mu_R = 2\mu_A$$

The potential constants were determined from a fit to the low energy parameters and the phase shifts up to 300MeV lab energy.

Using proton-proton scattering data and bound state of the neutron system, potential representing the interaction between two nucleons can be constructed. Although they ignore the important spin degree of freedom, the Malfliet-Tjon potentials are very useful in determining qualitative feature of nucleon system and providing test case.

Table 1 The various parameters sets of the Malfliet-Tjon potentials

Potential no.	$\mathbf{v_A}$ $\hbar c$	$\mathbf{u_A}$ **fm**	$\mathbf{v_R}$ $\hbar c$	$\mathbf{u_R}$ **fm**
I ($^1S_0$)	2.6	1.55	7 .39	3.11
II ($^1S_0$)	0.266	0.809	0	0
III ($^3S_1$)	3.22	1.55	7.39	3.11
IV ($^3S_1$)	0.330	0.633	0	0

The Lippmann-Schwinger Equation

The Lippmann-Schwinger equation is actually Schrödinger equation plus boundary condition independent of particular representation. The Lippmann-Schwinger equation is an exact equation for the scattering problem that can be expressed in momentum space as well as in configuration space. The Lippmann-Schwinger equation is the following equation:

$$\left|\Psi_{p_0}^{(+)}\right\rangle = \left|p_0\right\rangle + \frac{1}{E - \hat{H}_0 + i\varepsilon}\hat{V}\left|\Psi_{p_0}^{(+)}\right\rangle \tag{1}$$

Now by operating with V from the left hand side of equation (1)

$$\hat{V}\left|\Psi^{(+)}\right\rangle = \hat{V}\left|\Phi\right\rangle + \hat{V}\frac{1}{E - \hat{H}_0 + i\varepsilon}\hat{V}\left|\Psi^{+}\right\rangle \tag{2}$$

We define the transition operator $\hat{T}$ with respect to the plane wave state.

$$\hat{V}\left|\Psi^{(+)}\right\rangle = \hat{T}\left|\Phi\right\rangle \tag{3}$$

Φ is the plane wave. The operator $\hat{T}$ is defined so that the effect of $\hat{V}$ on Ψ^+ is the same as the effect of $\hat{T}$ on Φ.

And then equation (2) is become the following equation:

$$\hat{T}|\Phi\rangle = \hat{V}|\Phi\rangle + \hat{V}\frac{1}{E-\hat{H}_0+i\varepsilon}\hat{T}|\Phi\rangle \tag{4}$$

An operator equation for $\hat{T}$ is

$$\hat{T} = \hat{V} + \hat{V}\frac{1}{E-\hat{H}_0+i\varepsilon}\hat{T} \tag{5}$$

$$\hat{T} = \hat{V} + \hat{V}G_0\hat{T} \tag{6}$$

Where $$G_0 = \frac{1}{E-\hat{H}_0+i\varepsilon}$$

Then we project it onto the momentum space as follows:

$$\langle p|\hat{T}|p'\rangle = \langle p|\hat{V}|p'\rangle + \int p''^2 dp'' \langle p|\hat{V}G_0|p''\rangle\langle p''|\hat{T}|p'\rangle \tag{7}$$

By inserting the completeness relation in the second term of the above equation, the following equation is obtained.

$$\langle p|\hat{T}|p'\rangle = \langle p|\hat{V}|p'\rangle + \int p''^2 dp'' \frac{1}{E+i\varepsilon-\frac{p''^2}{2m}}\langle p|\hat{V}|p''\rangle\langle p''|\hat{T}|p'\rangle \tag{8}$$

Here E is the incident energy and it has $\frac{p_0^2}{2m}$, the Eq.(8) becomes

$$\langle p|\hat{T}|p\rangle = \langle p|\hat{T}|p'\rangle + 2m\int p''dp'' \frac{1}{p_0^2+i\varepsilon-p''^2}\langle p|\hat{V}|p''\rangle\langle p''|\hat{T}|p'\rangle \tag{9}$$

Then $2m\int p''^2 dp'' \frac{p_0^2}{p_0^2+i\varepsilon-p''^2}\langle p|\hat{V}|p_0\rangle\langle p_0|\hat{T}|p_0\rangle$ is added and subtracted into the second term of the above equation.

$$\begin{aligned}\langle p|\hat{T}|p'\rangle = \langle p|\hat{V}|p'\rangle &+ 2m\int p''^2 dp'' \frac{1}{p_0^2+i\varepsilon-p''^2}\langle p|\hat{V}|p''\rangle\langle p''|\hat{T}|p'\rangle \\ &+ 2m\int p''^2 dp'' \frac{p_0^2}{p_0^2+i\varepsilon-p''^2}\langle p|\hat{V}|p_0\rangle\langle p_0|\hat{T}|p_0\rangle \\ &- 2m\int p''^2 dp'' \frac{p_0^2}{p_0^2+i\varepsilon-p''^2}\langle p|\hat{V}|p_0\rangle\langle p_0|\hat{T}|p_0\rangle\end{aligned} \tag{10}$$

Using the principal value theorem $\lim_{i\varepsilon\to 0}\frac{1}{x+i\varepsilon} = \frac{p}{x} - i\pi\delta(x)$, the Eq.(10) becomes

$$\langle p|\hat{T}|p'\rangle = \langle p|\hat{V}|p'\rangle + 2m\int dp''p''^2 \frac{\langle p|\hat{V}|p''\rangle\langle p''|\hat{T}|p_0\rangle - p_0^2\langle p|\hat{V}|p_0\rangle\langle p_0|\hat{T}|p_0\rangle}{p_0^2 - p''^2} + mp_0\langle p|\hat{V}|p_0\rangle\langle p_0|\hat{T}|p_0\rangle\left[\ln\left|\frac{p_0+p_{\max}}{p_{\max}-p_0}\right| - i\pi\right] \tag{11}$$

The integral limit is zero to infinity but we determine p_{max} is enough for that limit and the saturation for $p' = p_0$, the eq.(11) becomes

$$T(p,p_0) = V(p,p_0)2m\int dp''p''^2 \frac{V(p,p'')T(p'',p_0) - p_0^2V(p,p_0)T(p_0,p_0)}{p_0^2 - p''^2} + mp_0\langle p|\hat{V}|p_0\rangle\langle p_0|\hat{T}|p_0\rangle\left[\ln\left|\frac{p_0+p_{\max}}{p_{\max}-p_0}\right| - i\pi\right] \tag{12}$$

This equation can be solved numerically by using the FORTRAN code.

Numerical Calculation

To obtain the numerical value of the T-matrix, we transform the Eq.(12) into the discrete form. By using the Gauss numerical integration method, these equations can be written as

$$T(p,p_0) = V(p,p_0) + 2m\sum_j^N W_j p_j^2 \frac{V(p_i,p_j)T(p_j,p_0)}{p_0^2 - p_j^2} - 2m\sum_j^N W_j \frac{p_0^2V(p,p_0)T(p_0,p_0)}{p_0^2 - p_j^2} + mp_0\langle p|\hat{V}|p_0\rangle\langle p_0|\hat{T}|p_0\rangle\left[\ln\left|\frac{p_0+p_{\max}}{p_{\max}-p_0}\right| - i\pi\right] \tag{13}$$

Where W_j and p_j are the Gauss weight and Gauss point. The following notations are used to simplify the equation

$$\tilde{W}_j = \frac{2mW_j}{p_0^2 - p_j^2}$$

$$C_i = mp_0V(p_i,p_0)\left[\ln\left|\frac{p_0+p_{\max}}{p_{\max}-p_0}\right| - i\pi\right]$$

$$\tilde{C}_i = C_i - \sum_j^N \tilde{W}_j p_0^2 V$$

For FORTRAN CODE, the final equation is

$$T_i = V_i + \sum_j^N \tilde{W}_j p_j^2 V_{ij} T_j + \tilde{C}_j T_0 \tag{14}$$

Here the index i run from 0 to N and j runs from 1 to N. in general, the above equation can be written in the matrix form

$$\begin{bmatrix} \tilde{C}_0 - 1 & \tilde{W}_1 p_1^0 V_{01} & \dots \tilde{W}_N P_N^2 V_{0N} \\ \tilde{C}_1 & \tilde{W}_1 p_1^0 V_{11} - 1 & \dots \tilde{W}_N P_N^2 V_{1N} \\ \tilde{C}_2 & \tilde{W}_1 p_1^0 V_{21} & \dots \tilde{W}_N P_N^2 V_{2N} - 1 \\ \vdots & \vdots & \dots \\ \tilde{C}_N & \tilde{W}_1 p_1^0 V_{N1} & \dots \tilde{W}_N P_N^2 V_{NN} - 1 \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix} = \begin{bmatrix} -V_0 \\ -V_1 \\ -V_2 \\ \vdots \\ -V_N \end{bmatrix} \quad (15)$$

The above matrix is denoted as

$$\sum_{j=1} A_{ij} T_j = B_i$$

The Eq(15) is a set of 'N' equations in 'N' unknowns. We can solve this equation by using Gauss Elimination Method. And then we obtain the T-matrix elements.

Using T-matrix, we calculated the phase shift. The relation between T-matrix and phase shift is given the following equation:

$$S = 1 - i\, m\, \pi\, p_0\, T(p_0, p_0)$$

The phase shift $\delta = \frac{1}{2}\, tan^{-1}\left(\frac{Img(S)}{Re(S)}\right)$

Then the effective range and scattering length are calculated. We have the equation

$$k \cot \delta = \frac{-1}{a} + \frac{1}{2} r\, k^2$$

where δ is the phase shift of a function of energy E.

$$k = \sqrt{2mE}$$

The effective range 'r' and the scattering length 'a' do not depend on energy E.

For the two values of input energy, the following two equations is obtained

$$-\frac{1}{a} + \frac{1}{2} k^2(E_1) r_0 = k(E_1) \cot \delta(E_1)$$

$$-\frac{1}{a} + \frac{1}{2} k^2(E_2) r_0 = k(E_2) \cot \delta(E_2)$$

These equations can be written as the following matrix forms:

$$\begin{pmatrix} -1 & \frac{1}{2} k^2(E_1) \\ -1 & \frac{1}{2} k^2(E_2) \end{pmatrix} \begin{pmatrix} \frac{1}{a} \\ r_0 \end{pmatrix} = \begin{pmatrix} k(E_1) \cot \delta(E_1) \\ k(E_2) \cot \delta(E_2) \end{pmatrix}$$

This equation is two unknown two equations. Two unknowns 'a' and 'r' can be calculated by using Gauss elimination method.

Results and Discussions

T-matrix elements are obtained by solving the Lippmann Schwinger equation with Gauss elimination method. And then, the phase shifts for various incident energies are calculated by sing the T-matrix elements. We have calculated the phase shifts for the singlet ($^1S_0$) and the triplet ($^3S_1$) potentials of Yukawa type which are given in table (1). The potential I and II are the singlet ($^1S_0$) potentials and the potential III and IV are triplet ($^3S_1$) potentials. The results of the $^1S_0$ phase shift for potential I and potential II is given in table (2) and figure (1). The results of the $^3S_1$ phase shift for potential III and potential IV is given in table (3) and figure (2). We compare the calculated results and experimental results.

Our calculated values of scattering length for $^1S_0$ potential I and II are -23.78fm and-23.67fm. Therefore our calculated values are in good agreement with experiment value of -23.75. The negative scattering length implies that the system has no bound state. Our calculated results of scattering length for $^3S_1$ potential III and IV are 5.34fm and 5.50fm. Our calculated values are in good agreement with the experimental value of 5.423fm. The positive scattering length implies that a bound state exists. The results of effective range for $^1S_0$ potential I and II are 2.84fm and 2.88fm and these results are in good agreement with the experimental value of 2.73fm. Our calculated results of effective range for $^3S_1$potential III and IV are 1.87fm and 1.73fm and these results are in good agreement with the experimental value of 1.73fm.

Table 2 Neutron-proton scattering phase shift for singlet state ($^1S_0$)

Elab (MeV)	Phase shift(Deg)		
	Exp (Deg)	Potential I (Deg)	Potential II (Deg)
1	62.1	64.65	61.12
10	60.0	58.41	59.75
25	50.9	47.04	53.31
50	40.5	34.96	47.47
100	26.8	20.35	41.38
150	16.9	10.90	37.86
200	8.9	3.90	35.32

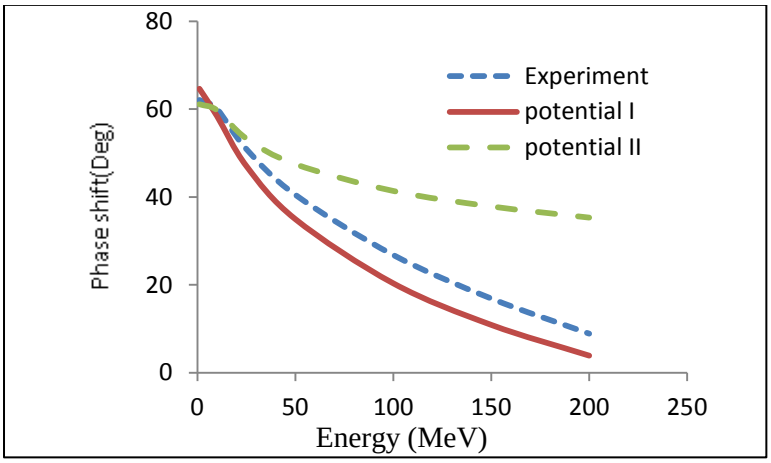

Figure 1 n-p phase shift for $^1S_0$ state

Table 3 Neutron-proton scattering phase shifts for triplet ($^3S_1$) state

Elab (MeV)	Phase shift(Deg)		
	Exp;	Potential III	Potential IV
24	81.7	79.83	86.03
48	63.3	61.77	73.25
96	42.9	43.17	61.74
144	29.8	32.10	55.57
208	17.1	21.99	50.35

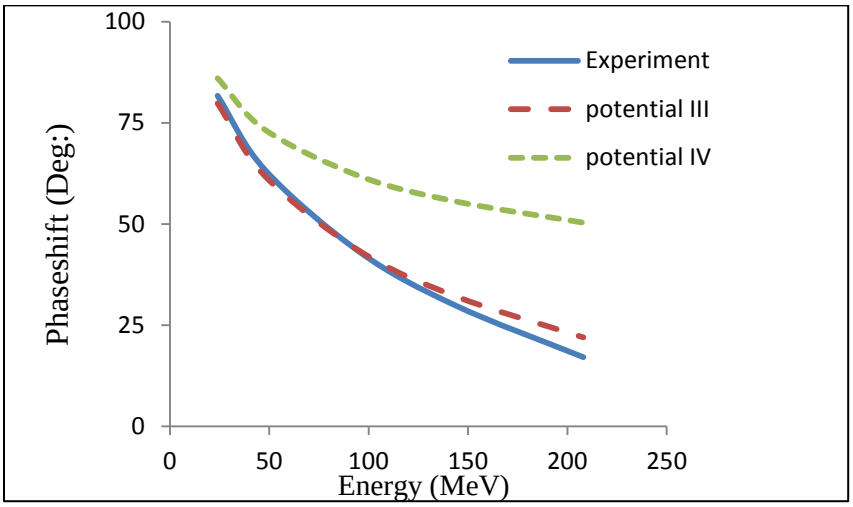

Figure 2 n-p phase shifts for triplet $^3S_1$ state

Table 4 The n-p scattering length and effective range for four parameter sets of Malfliet-Tjon potential

	$^1S_0$ state			$^3S_1$ state		
	I	II	Exp:[1]	III	IV	Exp:[1]
a (fm)	-23.78	-23.67	-23.715	5.34	5.50	5.423
r (fm)	2.84	2.88	2.73	1.87	1.73	1.73

Conclusion

The phase shifts for Yukawa type Malfliet-Tjon potential have been calculated. We obtained the T-matrix elements by solving the Lippmann-Schwinger equation with Gauss Elimination method. The phase shifts for various parameter sets of Malfliet-Tjon potential have been calculated. Potential II and IV are purely attractive potentials. The calculated phase shifts of Potential II and IV do not agree with the experimental results. Potential I and III are two-range potential including attractive and repulsive parts. Therefore, the results of phase shift for Potential I and Potential III are in good agreement with experimental results.

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