DYNAMICS AND SIMULATION OF THE GRAVITATIONAL COLLAPSE OF A NEUTRON STAR TO A BLACK HOLE

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Abstract

Using the highly non-linear differential equations, 3 +1 numerical relativity and Tolman-Oppenheimer-Volkoff equations, together with so called 1-log slicing condition (singularity avoiding gauge) it has been found that no real and coordinate singularity appears when the numerical simulations are carried out. Mass-Radius and Mass-Density visualization are implemented.

Keywords: Tolman-Oppenheimer-Volkoff equations, numerical simulations, Mass-Radius.

Introduction

Neutron stars are stellar remnants resulting from the gravitational collapse of a massive star during supernova event. Neutron stars are the most compact and smallest stars known to exist in the universe. Neutron stars are the end points of stellar evolution of massive stars whose corpse is not large enough to become a black hole. Black holes are even more compact than neutron stars: here, several solar masses are compressed within only a few kilometers "radius". They do not possess a distinct surface as such, but are delimited by a so-called event horizon which is the limit beyond which light cannot escape to infinity. General relativity is taken part in an important role in the formation of black holes and very important for the neutron stars. So the numerical simulations of gravitational collapse of rotating stellar configurations leading to the formation of black hole are a long standing problem in numerical relativity. This paper tries to simulate the collapse of a neutron star to a black hole. In the following the main theoretical concepts and basic equations required to understand the background of the gravitational collapsing of neutron star to black hole are discussed.

The Einstein Equations

The main theoretical concepts and the basic equations needed to understand the background of the gravitational collapse of neutron star to black hole are summarized. In connecting with the conservation laws for energy-momentum and rest-mass, Einstein's theory of general relativity is needed to solve the groundings of the differential equations. In highly nonlinear differential equation, the Einstein equations and the conservation laws are defined as follow

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}, \tag{1}$$

$$\nabla_{\mu} T^{\mu\nu} = 0, \tag{2}$$

$$\nabla_{\mu} \left(\rho u^{\mu} \right) = 0. \tag{3}$$

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Where $T_{\mu\nu}$ is the energy-momentum tensor, $R_{\mu\nu}$ is the Ricci tensor, which contains first and second derivatives of the space-time metric $g_{\mu\nu}$, ∇_μ is the covariant derivative and u^μ is the four velocity of the stars fluid. The Einstein equation describes in which way the space-time structure need to bend if energy-momentum is present. In this paper the energy-momentum, which curves space-time, arises from the large energy amount of the neutron star matter.

Tolman-Oppenheimer-Volkoff equations

The gravitational collapse of a neutron star to a black hole is depended on the equation of state (EOS), i.e. the relation between pressure and density in the neutron star interior (Lattimer & Pethick, 2004& Aaron Smith, 2012).To consider the mechanical structure of the neutron star is taken to be perfect fluid and spherical. For a perfect fluid, the energy momentum tensor is:

$$T^{\mu\nu} = \left(\rho + p\right) u^\mu u^\nu + p g^{\mu\nu} \tag{4}$$

The law of conservation of energy and momentum leads to

$$T^{\mu\nu}{}_{;\nu} = g^{\mu\nu}\partial_\nu p + \left[\left(\rho + p\right) u^\mu u^\nu\right]_{;\nu} + \left(\rho + p\right)\Gamma^{\mu}_{\nu\lambda} u^\nu u^\lambda = 0 \tag{5}$$

$$\nabla_\mu T^{\mu\nu} = 0,$$

For the spherically symmetric object, the space time is given by the Schwarzschild metric

$$ds^2 = -e^{2N(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{6}$$

where *N(r)* and *β(r)* are the metric functions depended on the radial coordinate r, t is the time component and *r, θ* and ϕ are the spatial component,

$$e^{2\beta(r)} = (1 - \frac{2Gm(r)}{r})^{-1} \tag{7}$$

and the *N(r)* can be expected from the following equation;

$$\frac{dN}{dr} = \frac{4\pi G p r^3 + Gm(r)}{r(r - 2Gm(r))} \tag{8}$$

By taking the boundary condition, $\lim_{r\to\infty} e^{-2N(r)} = \lim_{r\to\infty} e^{2\beta(r)} = 1$

So, $$e^{2N(r)} = (1 - \frac{2Gm(r)}{r}) \tag{9}$$

If $r = R_s = 2MG$ the Schwarzschild radius, $m \equiv m(r)$ is star radius, $m\left(r\right) = \int_0^R 4\pi r^2 \rho(r) dr$ is mass of the sphere with radius *r*. Then the structures of spherical symmetric stars are computed utilizing the Tolman-Oppenheimer-Volkoff relativistic structure equations (i.e. TOV equations):

$$\frac{dP}{dR} = -\frac{G(m(r) + 4\pi r^3 P)(\rho + P)}{r(r - 2Gm(r))} \tag{10}$$

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \tag{11}$$

Where *P* and *ρ* are the pressure and mass-energy density, and *m(r)* is the gravitational mass enclosed within a radius *r*. This connection is made by the equation of state of matter *P=P (ρ)* (the set of all spherical symmetric), in particular, an estimate for the maximum mass of the star can be obtained. These sets of nonlinear equations are for p(r) and m(r) from $r = 0$ for starting value of $p(r = 0) = p_0$ to the point $r = R$ where the pressure $p(r = R) = 0$. At that point R is the radius of the star. Since $p(r = R) = 0$, it is the vacuum outside of the star because the pressure gradually decreases outwards from the center. So there require that exterior metric should be Schwarzschild metric. Then the metric functions must be continuous at $r = R$:

Inside the star
$$g_{rr} = (1 - \frac{2m(r)}{r})^{-1} \tag{12}$$

and outside the star
$$g_{rr} = (1 - \frac{2M}{r})^{-1} \tag{13}$$

The total mass of the star is defined $M = m(r)$ (14)

Thus total mass of the star is determined by distant orbits, $M = \int_0^R 4\pi r^2 \rho(r) dr$ (15)

Outside the distribution of mass, which terminates at the radius of star *R*, there is in vacuum with $p(r = R) = 0$ and Einstein equations give

$$g_{00}(r = R) = -(1 - \frac{2M}{r}) \tag{16}$$

The important information which can be obtained by solving the TOV equation is the mass-radius relationship for a neutron star given a particular EOS model. If $r > R_s$, where *r* is the radius of star, and $R_s = 2MG$, i.e., the Schwarzschild radius, with the mass of the star *M* and the gravitational constant *G*. For $r < R_s$, the star becomes unstable and connected with gravitational collapse and likely to form a black hole. The 3-D variation of the gradient of the gravitational potential can be visualized in terms of radial function *r* and mass *m*.

The gravitational collapse of neutron star to black holes

To simulate the evolution of a collapse of a neutron star, the Einstein's equations are needed to reformulated and solve the time dependent problem numerically. This reformulation, the so called (3+1) split, starts by slicing the 4-dimensional manifold *M* into 3-dimensional space like hypersurface Σ_t. The space-time metric g_{ab} is then also divided into a purely spatial metric γ_{ij} and a lapse function *N* and a shift vector. Homogeneously, the line element may be written as

$$ds^2 = g_{ab} dx^a dx^b = -N^2 dt^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right) \tag{17}$$

where γ_{ij} and γ^{ij} is raise the lower indices of spatial tensors, *N* is the lapse function and β^i the shift vector and which is often referred to as the metric in 3+1 form(Baumgarte, 1998), here ds^2=-(proper time between neighboring spatial hypersurfaces)^2 +(proper distance between the spatial hypersurface)^2.

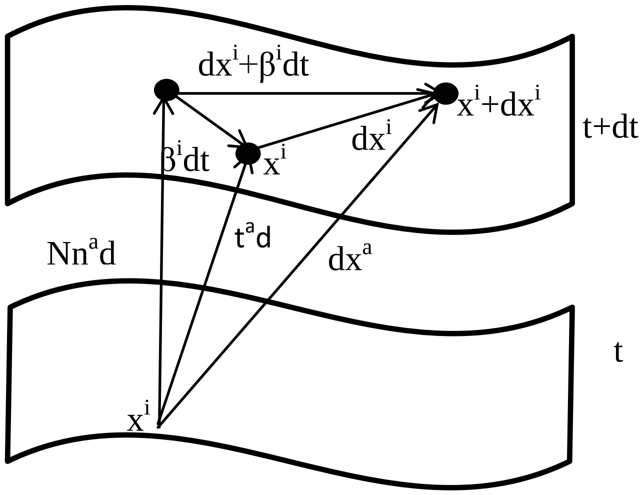

Figure 1 3+1 decomposition of spacetime

The coordinate label x^i moves through spacetime from one slice to another in a way given by the lapse N and shift β^i .Thus this equation can determine the invariant interval between neighboring points. The covariant components of metric g_{ab} are

$$g_{ab} = \begin{pmatrix} -N^2 + \beta^i \beta_i & \beta_i \\ \beta_i & \gamma_{ij} \end{pmatrix} \qquad (18)$$

The lapse function *N* describes the difference between the coordinate time t and the proper time of a fluid particle. The shift vector β_i measures how the coordinates are shifted on the spatial slice if the fluid particle moves an infinitesimal time step further in figure. By using the above matric, the first order differential equation called ADM equations are reformulated. The ADM equations have two set of equations called constraint equations and evolution equations: the constraint equations are

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho \text{ (Hamiltonian constraint)} \qquad (19)$$

and $$D_j K^{ji} - D_i K = 8\pi j^i \text{ (momentum constraint)} \qquad (20)$$

the evolution equations are

$$\partial_t \gamma_{ij} = -2NK_{ij} + D_i\beta_j + D_j\beta_i \qquad (21)$$

$$\partial_t K_{ij} = -D_i D_j N + N\left(R_{ij} + KK_{ij} - 2K_{ik}K^k{}_j\right) + 4\pi N\left(\gamma_{ij}\left(S-\rho\right) - 2S_{ij}\right)$$

$$+\beta^i D_k K_{ij} + K_{ik} D_j \beta^l + K_{kj} D_i \beta^k \qquad (22)$$

The shift terms in the last two equations arise from the Lie derivatives $\mathcal{L}_\beta \gamma_{ab}$ and $\mathcal{L}_\beta K_{ab}$. Here $\mathcal{L}_\beta$ is the lie derivative along the shift vector and that Ricci tensor R_{ij} is given be second spatial derivatives of the metric (Aaron Smith, 2012, Arnowitt et al., 1962, Arnowitt et al., 2008).

As a result this is not first order system. The constraint equations constraint the field variables on each spatial slice and the lapse and the evolution equations determine the time evolution of the fields from one spatial slice to the next.

Singularity avoidance conditions and 1- log slicing

In particularly, most of the modern 3+1 solution of the Einstein's equations adopt as hyperbole slicing condition a member of the so called Bona-Masso` family of slicing conditions(Bona C. et al, 2005), which can be generically written as

$$\left(\partial_t - \mathcal{L}_\beta\right) N = -KN^2 f\left(N\right) \tag{23}$$

where *f* is an arbitrary function and *f(N)>0*. By varying the expression of the generic function *f(N)*, the slicing condition recovers a number of well-known slicing. The geodesic slicing condition also fulfills this relation with *f = 0*. If by setting *f =q/N* with q is an integer, the slicing condition obtains the generalized "1- log" slicing condition where $N = h\left(x^i\right) + \ln\gamma^{1/2}$, $h(x_i)$ is positive but otherwise arbitrary time independent function. In practice, most numerical simulations set *f=2/N* lead to

$$\left(\partial_t - \mathcal{L}_\beta\right) N = -2KN \tag{24}$$

Substituting Eqn. (21) for *–KN*,

$$\left(\partial_t - \mathcal{L}_\beta\right) N = \partial_t \ln\gamma - 2D_i\beta^i \tag{25}$$

If normal coordinates are used, β=0, the above equation becomes

$$\partial_t N = \partial_t \ln\gamma \tag{26}$$

a solution of which is $$N = 1 + \ln\gamma \tag{27}$$

For this reason, a foliation whose lapse function obeys is called a 1-log slicing. Which has been shown to be very robust and well behaved not only in vacuum spacetimes representing black holes but also in spacetimes describing the neutron stars.

A coordinate system can be constructed by identifying the time coordinate vector with the Killing vector ξ^a, so

$$\xi^a = t^a = N_K n^a + \beta_K^a \tag{28}$$

N_K is called Killing lapsed and β_K^a is called Killing shift. For Schwarzschild spacetime, the foliated by slices of constant Schwarzschild time t, the Killing lapse can be identified

$$N_K = \frac{1 - M/(2r)}{1 + M/(2r)} \tag{29}$$

and the Killing shift as $\beta_K^r = 0$. But N_K is negative for *r >M/2*. Having the slicing lapse equal to the Killing lapse, $\partial_t N_K = \partial_t N_S = 0$ the slicing shift is equal to the Killing shift, then condition (24) reduces to

$$K = \frac{\beta_K^i \partial_i N_K}{2N_K} \tag{30}$$

This condition is stationary 1-log slicing. This condition can be employed for the construction of initial data. Besides this condition considered without the adventive term,

$$\partial_t N = -2NK \tag{31}$$

In this case the Killing lapse associated with a stationary slicing satisfied this condition only if the slices are maximal, i.e., if *K=0*. For Schwarzschild, the maximal slicing can be parameterized by a parameter C, the lapse is yield

$$N_K = (1 - \frac{2M}{R} + \frac{C^2}{R^4})^{1/2} \tag{32}$$

where *R* is an areal radius. For *C=0*, it give the slice of constant Schwarzschild time t. When applying the 1-log slicing, the right hand side *-2NK* is replaced by *–n f (N) K*, where n is some arbitrary number and *f (N)* some non-zero and finite function of *N*. Besides, 1+log slicing condition can be applied to binaries.

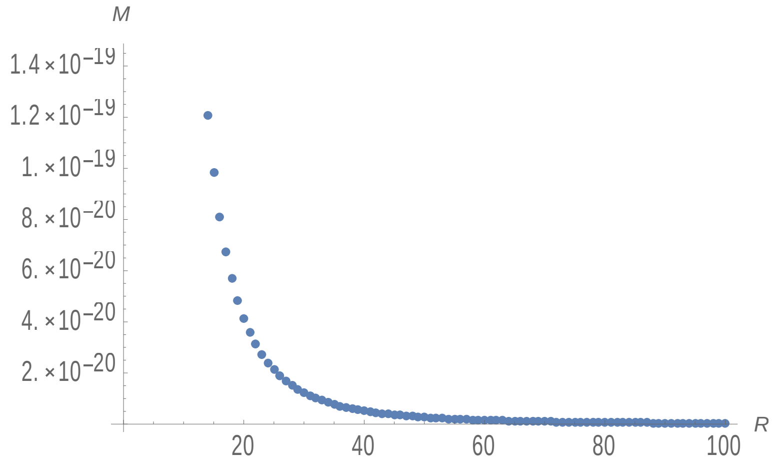

Figure 2 Mass-Radius relation of neutron star

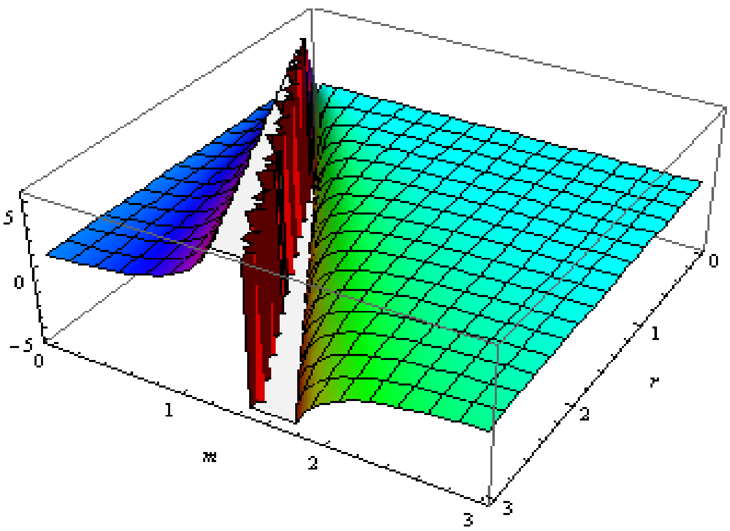

Figure 3 3D visualization of the gravitational potential with radius r and mass m

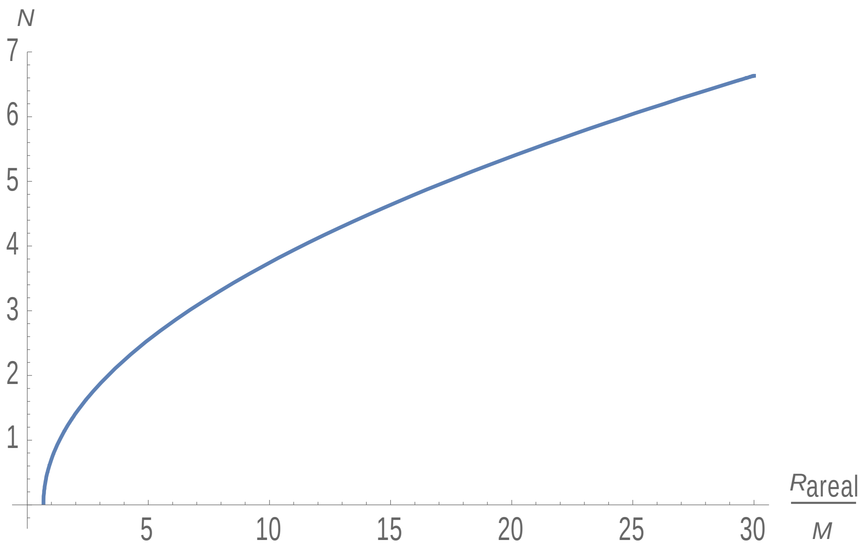

Figure 4 Graph of the lapse N for Schwarzschild in 1-log slicing

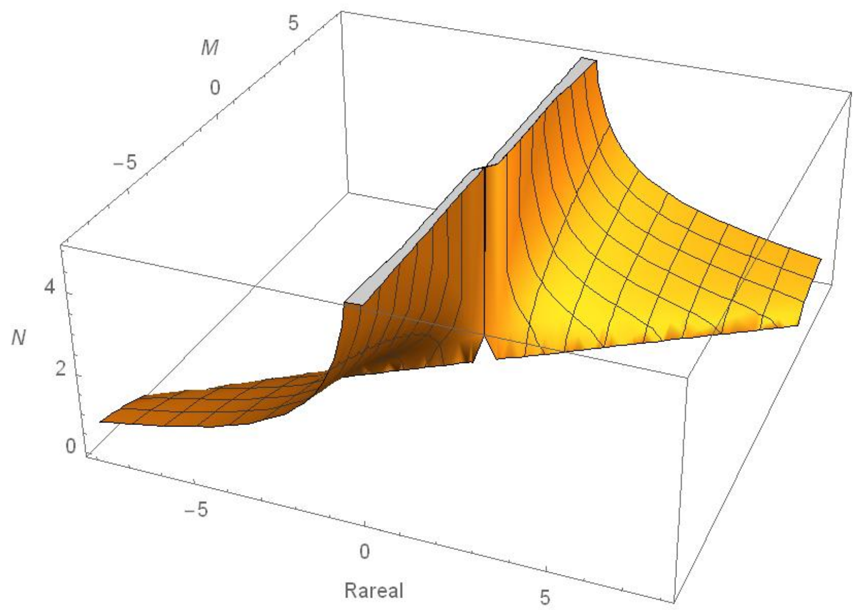

Figure 5 3D visualization of the lapse N in the 1-log slicing

Concluding Remarks

When the central density of a neutron star exceeds the maximal allowed value , the star collapses to a black hole and its radius shrinks until it reaches the value of the location of the event horizon of the corresponding black hole at r = 2M. . If $r > R_s$, where r is the radius of star, for $r < R_s$, the star becomes unstable and connected with gravitational collapse and likely to form a black hole and at $R_s = 0$ and R=2M (i.e., the Schwarzschild radius, with the mass of the star M), singularity point are formed. So, Figure 2 shows the relationship between mass and radius of the neutron star. Figure 3 show 3D visualization of the gravitational potential with radius r and mass m. To slow down the evolution near the real singularity, a singularly avoiding coordinates so called "1-log" slicing has been used fairly commonly. No real or coordinate singularity could appear in this gauge during the numerical evolution. Then, Figure 4 shows the graph of the lapse N for Schwarzschild in the 1-log slicing. When applying the 1-log slicing, Figure 5 shows 3D visualization of the lapse N in the 1-log slicing, which would avoid the singularity point where the gravitational fields are strongest.

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