J. Myanmar Acad. Arts Sci. 2020 **Vol. XVIII.No.2B**

DERIVATIONS AND VISUALIZATIONS OF COSMOLOGICAL FLUID EQUATIONS

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Abstract

The fundamental notion of fluid cosmology has been briefly studied. In order to derive various physical and geometrical aspects of mass, momentum, density and equation of state, it is essentially needed to make use of the governing equation frequently used in astrophysics and cosmology. Then, it has been known clearly in this research. Some interesting and informative visualization of the results are done with the help of Mathematica Software.

Keywords*: geometrical aspects of mass, momentum, density and equation of state*

Introduction

Fluid dynamics is one of the most important of all areas of physics life as we know it would not exist without fluids. The air we breathe and the water we drink (and which makes up most of our body mass) are fluids. Motion of air keeps us comfortable in a warm room, and air provides the oxygen we need to sustain life. Similarly, most of our (liquid) body fluids are water based. It is clear that fluids are completely necessary for the support of carbon based life forms. [Brown, J. D., etal (1994)]

The three main approaches to the study of fluid dynamics:

(i) Theoretical,

(ii) Experimental and

(iii) Computational.

Fluid Properties

There are two main classes of fluid properties: transport properties and physical properties. Three basic transport properties are viscosity, thermal conductivity and mass diffusivity. The physical properties are density and pressure, both of which might be viewed as thermodynamic properties, especially in the context of fluids. [Batchelor George, K., (1967)]

The viscosity is a measure of the resistance of a fluid which is being deformed by either shear stress or tensile stress. In everyday terms viscosity is "thickness" or "internal friction". Viscosity describes a fluid's internal resistance to flow and may be thought of as a measure of fluid friction with the exception of super fluids, all real fluids have some resistance to stress and therefore are viscous.

Thermal conductivity is the transport property that mediates diffusion of heat through a substance in a manner analogous to that already discussed in considerable detail with respect to viscosity and momentum. Thermal energy, so thermal conductivity provides an indication of how quickly thermal energy diffuses through a medium. Fourier’s law of heat conduction,

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$$q = -k\frac{dT}{dy} \tag{1}$$

where, q = heat flux.

k = thermal conductivity.

$\frac{dT}{dy}$ = component of the temperature gradient in the y direction

The Fundamental Fluid Dynamics Equations

The equations of fluid dynamics are best expressed via conservation laws for the conservation of mass, momentum and energy.

Conservation of Mass

We consider the rate of change of mass within a fixed volume. This the changes as a result of the mass flow through the bounding surface.

$$\frac{\partial}{\partial t}\int_v \rho dV = -\int_s \rho V_i n_i ds \tag{2}$$

Using the divergence theorem,

$$\frac{\partial}{\partial t}\int_V \rho dV + \int_V \frac{\partial}{\partial x_i} \rho V_i n_i ds = 0 \tag{3}$$

$$\int_{\boldsymbol{V}} [\frac{\boldsymbol{\partial\rho}}{\boldsymbol{\partial t}} + \frac{\partial}{\partial x_i}(\rho V_i)]dV = 0 \tag{4}$$

The continuity equation, since the volume is arbitrary,

$$\frac{\boldsymbol{\partial\rho}}{\boldsymbol{\partial t}} + \frac{\partial}{\partial x_i}(\rho V_i) = 0 \tag{5}$$

Conservation of Momentum

We consider the rate of change of momentum within a volume. This decreases as a result of the flux of momentum through the bounding surface and increases as the result of body forces (in our case, gravity) acting on the volume.

Let Π_{ij} = Flux of component of momentum in the direction

f_i = Body force per unit mass

then,

$$\frac{\partial}{\partial t}\int_V \rho V_i dV = -\int_S \Pi_{ij}\, n_i\, dS + \int_V \rho f_i \,\mathrm{dV} \tag{6}$$

There is an equivalent way of thinking of Π_{ij} , which is often useful, and that is $\Pi_{ij}\, n_i\, dS$, is the ith component of the force exerted on the fluid exterior to S by the fluid interior to S.

Again using the divergence theorem,

$$\int_V [\frac{\partial}{\partial t}(\rho V_i) + \frac{\partial \Pi_{ij}}{\partial x_j}]dV = \int_V \rho f_i \,\mathrm{dV} \tag{7}$$

$$\frac{\partial}{\partial t}(\rho V_i) + \frac{\partial \Pi_{ij}}{\partial x_j} = \rho f_i \tag{8}$$

For gravity we use the gravitational potential

$$f_i = -\frac{\partial \emptyset_G}{\partial x_i} \tag{9}$$

For a single gravitating object of mass M,

$$\emptyset_G = -\frac{GM}{r} \tag{10}$$

And for a self-gravitating distribution,

$$\nabla^2 \emptyset_G = 4\pi G\rho \tag{11}$$

$$\emptyset_G = -G\int_{V'} \frac{\rho(x_i)}{|x_i - x_i'|} d^3x' \tag{12}$$

Where G is Newton's constant of gravitation.

The momentum flux is composed of a bulk part plus a part resulting from the motion of particles moving with respect to the center of mass velocity of fluid (v_i). For a perfect fluid, we take ρ to be the isotopic pressure, then,

$$\Pi_{ij} = \rho V_i V_j + p\delta_{ij} \tag{13}$$

The equations of motion are then,

$$\frac{\partial}{\partial t}(\rho V_i) + \frac{\partial \Pi_{ij}}{\partial x_j} = \rho f_i \tag{14}$$

$$\frac{\partial}{\partial t}(\rho V_i) + \frac{\partial}{\partial x_j}\left(\rho V_i V_j + \rho\delta_{ij}\right) = -\rho\frac{\partial \emptyset_{ij}}{\partial x_i} \tag{15}$$

$$\frac{\partial}{\partial t}(\rho V_i) + \frac{\partial}{\partial x_j}\left(\rho V_i V_j\right) = -\rho\delta_{ij} - \rho\frac{\partial \emptyset_{ij}}{\partial x_i} \tag{16}$$

There is also another useful form for the momentum equation derived using the continuity equation

$$\frac{\partial}{\partial t}(\rho V_i) + \frac{\partial}{\partial x_i}\left(\rho V_i V_j\right) = V_i\frac{\partial \rho}{\partial t} + \rho\frac{\partial V_i}{\partial t} + V_j\frac{\partial}{\partial x_j}\left(\rho V_j\right) + \rho V_j\frac{\partial V_i}{\partial V_j} \tag{17}$$

$$= -V_i[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho V_j\right) + [\rho\frac{\partial V_i}{\partial t} + \rho V_j\frac{\partial V_i}{\partial V_j}] \tag{18}$$

$$\frac{\partial}{\partial t}(\rho V_i) + \frac{\partial}{\partial x_i}\left(\rho V_i V_j\right) = \rho\frac{\partial V_i}{\partial t} + \rho V_j\frac{\partial V_i}{\partial V_j} \tag{19}$$

Hence, another form of the momentum equation is

$$\rho\frac{\partial V_i}{\partial t} + \rho V_j\frac{\partial V_i}{\partial x_j} = -\frac{\partial \rho}{\partial x_i} - \rho\frac{\partial \emptyset_G}{\partial x_i} \tag{20}$$

On dividing by the density,

$$\frac{\partial V_i}{\partial t} + V_j\frac{\partial V_i}{\partial x_i} = -\frac{1}{\rho}\frac{\partial \rho}{\partial x_i} - \frac{\partial \emptyset_G}{\partial x_i} \tag{21}$$

Conservation of Energy

We take the momentum equation in the form,

$$\rho \frac{\partial V_i}{\partial t} + \rho\, V_j \frac{\partial V_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} - \rho \frac{\partial \emptyset_G}{\partial x_i} \tag{22}$$

Take the scalar products with the velocity,

$$\rho V_i \frac{\partial V_i}{\partial t} + \rho\, V_j V_i \frac{\partial V_i}{\partial x_j} = -V_i \frac{\partial p}{\partial x_i} - \rho V_i \frac{\partial \emptyset_G}{\partial x_i} \tag{23}$$

$$\rho \frac{\partial}{\partial t}[\tfrac{1}{2} V_i V_i] + \rho\, V_j \frac{\partial}{\partial x_j}[\tfrac{1}{2} V_i V_i] = -V_i \frac{\partial p}{\partial x_i} - \rho V_i \frac{\partial \emptyset_G}{\partial x_i} \tag{24}$$

$$\rho \frac{\partial}{\partial t}[\tfrac{1}{2} V^2] + \rho\, V_j \frac{\partial}{\partial x_j}[\tfrac{1}{2} V^2] = -V_i \frac{\partial p}{\partial x_i} - \rho V_i \frac{\partial \emptyset_G}{\partial x_i} \tag{25}$$

Before we use the continuity equation to move the ρ and ρV_ioutsides the differentiations. Now we can use the same technique to move them inside and we recover the equation.

$$\frac{\partial}{\partial t}[\tfrac{1}{2}\, \rho\, V^2] + \frac{\partial}{\partial x_j}[\tfrac{1}{2} \rho\, V_j V^2] = -V_i \frac{\partial p}{\partial x_i} - \rho V_i \frac{\partial \emptyset_G}{\partial x_i} \tag{26}$$

The aim of the following is to put the right hand side into some sort of divergence form. We consider the first term.

$$-V_i \frac{\partial p}{\partial x_i} = \rho k T V_i \frac{\partial s}{\partial x_i} - \rho V_i \frac{\partial h}{\partial x_i} \tag{27}$$

$$-V_i \frac{\partial p}{\partial x_i} = \rho k T V_i \frac{ds}{dt} - \rho k T \frac{\partial s}{\partial t} - \rho V_i \frac{\partial}{\partial x_i} \tag{28}$$

$$= \rho k T \frac{ds}{dt} - \frac{\partial \varepsilon}{\partial t} + h \frac{\partial \rho}{\partial t} - \rho V_i \frac{\partial h}{\partial x_i} \tag{29}$$

We now eliminate the $\frac{\partial \rho}{\partial t}$ term suing continuity.

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_i}(\rho V_i) \tag{30}$$

And we obtain

$$-V_i \frac{\partial p}{\partial x_i} = \rho k T \frac{ds}{dt} - \frac{\partial \varepsilon}{\partial t} + h \frac{\partial}{\partial x_i} - \rho V_i \frac{\partial h}{\partial x_i} \tag{31}$$

$$-V_i \frac{\partial p}{\partial x_i} = \rho k T \frac{ds}{dt} - \frac{\partial \varepsilon}{\partial t} - \frac{\partial}{\partial x_i}(\rho h V_i) \tag{32}$$

The term

$$-\rho V_i \frac{\partial \emptyset_G}{\partial x_i} = -\frac{\partial}{\partial x_i}(\rho \emptyset_G V_i) + \emptyset_G \frac{\partial}{\partial x_i}(\rho V_i) \tag{33}$$

$$= -\frac{\partial}{\partial x_i}(\rho \emptyset_G V_i) + \emptyset_G \frac{\partial \rho}{\partial x_i} \tag{34}$$

$$= -\frac{\partial}{\partial x_i}(\rho \emptyset_G V_i) - \frac{\partial}{\partial t}(\rho \emptyset_G) + \rho \frac{\partial \emptyset}{\partial x_t} \tag{35}$$

When the gravitational potential is constant in time, $\frac{\partial \emptyset_G}{\partial t} = 0$

$$-\rho V_i \frac{\partial \emptyset}{\partial x_i} = -\frac{\partial}{\partial x_i}(\rho \emptyset_G V_i) - \emptyset_G \frac{\partial}{\partial x_i}(\rho \emptyset_G) \tag{36}$$

Hence the energy equation,

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho V^2\right]+\frac{\partial}{\partial x_i}\left[\frac{1}{2}\rho V^2 V_j\right]=-V_i\frac{\partial p}{\partial x_i}-\frac{\partial \emptyset_G}{\partial x_i}(\rho V_i) \quad (37)$$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho V^2\right]+\frac{\partial}{\partial x_i}\left[\frac{1}{2}\rho V^2 V_j\right]=\rho kT\frac{ds}{dt}-\frac{\partial \varepsilon}{\partial t}-\frac{\partial}{\partial x_i}(\rho h V_i)-\frac{\partial h}{\partial x_i}(\rho \emptyset_G V_i)-\frac{\partial}{\partial t}(\rho\emptyset_G) \quad (38)$$

Bringing terms over to the left hand side;

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho V^2+\varepsilon+\rho\emptyset_G\right]+\frac{\partial}{\partial x_j}\left[\frac{1}{2}\rho V^2 V_j+\rho h V_j+\rho\emptyset_G V_j\right]=\rho kT\frac{ds}{dt} \quad (39)$$

When the fluid is adiabatic

$$\rho kT\frac{ds}{dt}=0 \quad (40)$$

And we have the energy equation for a perfect fluid

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho V^2+\varepsilon+\rho\emptyset_G\right]+\frac{\partial}{\partial x_j}\left[\frac{1}{2}\rho V^2 V_j+\rho h V_j+\rho\emptyset_G V_j\right]=0 \quad (41)$$

The total energy per unit volume is

$$E=\frac{1}{2}\rho V^2+\varepsilon+\rho\emptyset_G \quad (42)$$

$$\varepsilon \coloneqq 1;\emptyset_G:=4$$

Show [Plot 3D [(1/2 ρ v^2+ε +$\rho\emptyset_G$), { ρ,-8 Pi,8Pi},{v,-6 Pi,6 Pi},Plot Points $\rightarrow$ 35,

Axes $\longrightarrow$True, Axes Label $\rightarrow$ {“ρ”, “v”, “ε”}, Display Function $\rightarrow$ $ Display Function]]

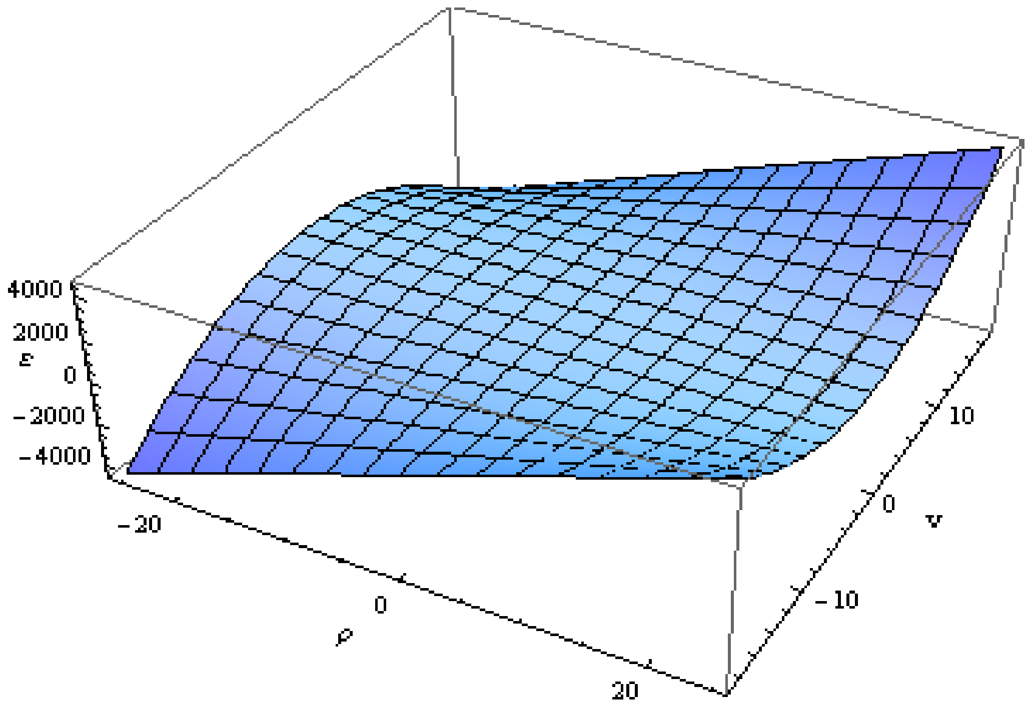

Figure 1 3D variation of energy with density and velocity.

Thermodynamic and its physical Properties

Thermodynamic

The thermodynamics of a volume element of fluid and the variables used to describe its state as

m=mass of element

ε = internal energy density per unit volume

p =pressure

S = entropy per unit mass

T =temperature (in degree Kelvin)

The second law of thermodynamic tells us that the change in entropy of mass of a gas is related to change in other thermodynamic variables as follows;

$$kTdS = dU + pdU \tag{43}$$

$$kTd(ms) = d\frac{m\varepsilon}{\rho} + pd\frac{m}{\rho} \tag{44}$$

$$kTds = d\ \left(\frac{\varepsilon}{\rho}\right) + pd(\frac{1}{\rho})$$

$$= \frac{1}{\rho}d\varepsilon - \left(\frac{\varepsilon+p}{\rho^2}\right)d\rho \tag{45}$$

Expanding the differentials and multiply by ρ ,

$$\rho kTds = d\varepsilon - \left(\frac{\varepsilon+p}{\rho^2}\right)d\rho \tag{46}$$

In terms of derivatives along the trajectory of the element

$$\rho kT\frac{ds}{dt} = \frac{d\varepsilon}{dt} - \left(\frac{\varepsilon+p}{\rho^2}\right)\frac{d\rho}{dt} \tag{47}$$

The equation of continuity

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho V_i) = 0 \tag{48}$$

$$\frac{\partial\rho}{\partial t} + V_i\frac{\partial\rho}{\partial x_i} + \rho\frac{\partial v_i}{\partial x_i} = 0 \tag{49}$$

can be expressed in the form

$$\frac{\partial V_i}{\partial x_i} = -\frac{1}{\rho}\frac{d\rho}{dt} \tag{50}$$

So that

$$\rho kT\frac{ds}{dt} = \frac{d\varepsilon}{dt} - (\varepsilon + p)\frac{\partial V_i}{\partial x_i} \tag{51}$$

Enthalpy

The care of an isotropic distribution function (perfect fluid) introduces the enthalpy density defined by

$$H = \varepsilon + p \tag{52}$$

And the specific enthalpy (enthalpy per unit mass),

$$h = \frac{\varepsilon+p}{\rho} \tag{53}$$

for a monatomic gas,

$$\varepsilon = \frac{3}{2}nkT \tag{54}$$

$$p = nkT \tag{55}$$

And $\rho = \mu n m_p$

Where μ the mean is molecular weight and m_p is the mass of a proton. Hence,

$$h = \frac{(\frac{3}{2}kT+nkT)}{\mu n m_p} \tag{56}$$

$$h = \frac{5}{2}\frac{kT}{\mu m_p} \tag{57}$$

using the specific enthalpy, the energy flux,

$$F_E = (\rho\frac{1}{2}V^2 + h + \emptyset)V_i \tag{58}$$

Specific Enthalpy

A commonly used thermodynamic variable is the specific enthalpy,

$$h = \frac{\varepsilon+p}{\rho} \tag{59}$$

In terms of the specific enthalpy, the equation,

$$kTds = d\left(\frac{\varepsilon}{\rho}\right) + pd(\frac{1}{\rho}) \tag{60}$$

becomes,

$$kTds = d\left(\frac{\varepsilon+p}{\rho}\right) - d(\frac{p}{\rho}) + pd(\frac{1}{\rho}) \tag{61}$$

$$kTds = dh - \frac{dp}{\rho} \tag{62}$$

After dividing by the time increment of a volume element,

$$\rho kT\frac{ds}{dt} = \frac{d\varepsilon}{dt} - \frac{(\varepsilon+p)}{\rho}\frac{d\rho}{dt} \tag{63}$$

$$kT\frac{ds}{dt} = \frac{dh}{dt} - \frac{1}{\rho}\frac{d\rho}{dt} \tag{64}$$

The fluid is adiabatic when there is no transfer of heat in or out of the volume element,

If $$kT\frac{ds}{dt} = 0 \tag{65}$$

$$\frac{d\varepsilon}{dt} - \frac{(\varepsilon+p)}{\rho}\frac{d\rho}{dt} = 0 \qquad (66)$$

$$\frac{dh}{dt} - \frac{1}{\rho}\frac{d\rho}{dt} = 0 \qquad (67)$$

The equation ds,dε, dp are prefect differentials and these relationships are valid relations from point to point within the fluid two particular relationships we shall use in the following are

$$\rho kT\frac{ds}{dt} = \frac{d\varepsilon}{dt} - h\frac{\partial\rho}{\partial t} \qquad (68)$$

$$\rho kT\frac{\partial s}{\partial x_i} = \rho\frac{\partial h}{\partial x_i} - \frac{\partial\rho}{\partial x_i} \qquad (69)$$

Equation of State

Consider the following form of the entropy, internal energy, pressure relation;

$$\rho kTds = d\varepsilon - \frac{(\varepsilon+p)}{\rho^2}d\rho \qquad (70)$$

$$p = \frac{\rho kT}{\mu m_p} \qquad (71)$$

Where,μ is the mean molecular weight.

$$p = (\gamma - 1)\varepsilon \qquad (72)$$

$$\varepsilon + p = \gamma\varepsilon$$

hence, $$\mu m_p(\gamma - 1)\varepsilon ds = d\varepsilon - \frac{\gamma\varepsilon}{\rho}d\rho \qquad (73)$$

$$\mu m_p(\gamma - 1)\varepsilon ds = \frac{d\varepsilon}{\varepsilon} - \frac{\gamma}{\rho}d\rho \qquad (74)$$

$$\mu m_p(\gamma - 1)(s - s_0) = ln\varepsilon - \gamma ln\rho \qquad (75)$$

$$\frac{\varepsilon}{\rho r} = \exp[\mu m_p(\gamma - 1)(s - s_0)] \qquad (76)$$

$$= \exp[\mu m_p(\gamma - 1)s] \qquad (77)$$

We can discard s_0 since the origin of entropy arbitrary. Therefore,

$$\varepsilon = \exp[\mu m_p(\gamma - 1)s] \times \rho\gamma \qquad (78)$$

$$p = (\gamma - 1)\exp[\mu m_p(\gamma - 1)s] \times \rho\gamma \qquad (79)$$

$$p = k(s)\rho\gamma \qquad (80)$$

The function k(s) is often referred to as the pseudo-entropy. For a completely ionized monatomic gas $\gamma = \frac{5}{3}$.

Table[Plot3D[Bessel1J[0,((ε+p)/ρ)],{ ρ,-2,2},{p,1,5}, Plot Range→{-0.5,1}, Plot Label →Row[{ "ε=", padded Form[1* ε,1]}],Axes →True, Axes Label→ {" ρ", "p", " ε " },Displayed Function Identity],{ ε,1,6}]//Short

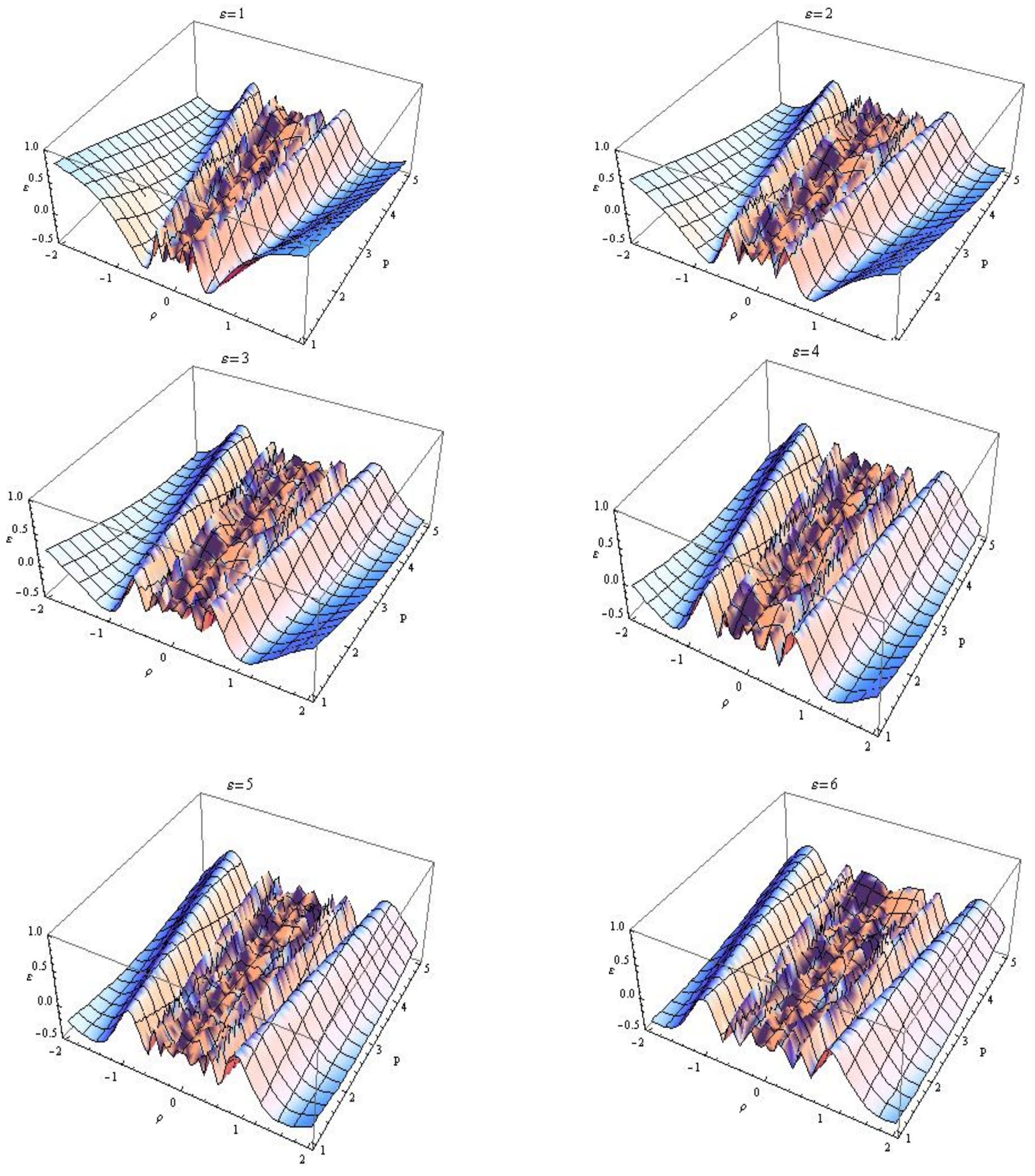

Figure 2 3D animated plot of specific enthalpy against density and pressure with ε from 1 to 6

Conclusion

We have attempted to provide an introduction to basic physics of fluid. The other various fluid properties, most of which are familiar with elemental physics and thermodynamics are carried out. Detailed computations of conservations of mass, momentum and energy are also presented. It is continued to probe the derivation of equation of state in very simple way. Next, the specific enthalpy which is commonly used in thermodynamics for the pressure or some other thermodynamic variable and the most natural is the entropy.

Acknowledgements

The author acknowledges the support of Professor Dr Khin Khin Win, Head of Department of Physics, University of Yangon. I wish to express I would like to express my deepest gratitude to my respectful Rector, Dr Kyaw Tun, Pro-rector Dr Yaw Han Htun and Pro-rector Dr Hla Hla Tin, Lashio University, for giving me a chance to write this research paper. Also, I would like to acknowledge Professor Dr Ko Ko Naing, Professor and head of Physics Department, Lashio University, for his valuable suggestion and comments for this paper.

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