J. Myanmar Acad. Arts Sci. 2018 **Vol. XVI. No.3**

RELATIONS BETWEEN CAYLEY GRAPH AND VERTEX-TRANSITIVE GRAPH

Aye Aye Myint*

Abstract

In this paper, we first express basic concepts of graph theory. Then we define vertex-transitive graph and Cayley graph with a given group by using generating set or nongenerating set. Finally, we prove that every Cayley graph is vertex-transitive graph and we also give an example that the converse of this theorem is false.

Keywords: graph, digraph, connected, vertex-transitive, group, order, Cayley graph, Cayley digraph, diameter of a graph.

1. Basic Concepts of Graph Theory

A **graph** G = (V(G), E(G)) with n vertices and m edges consists of a vertex set $V(G)=\{ v_1, v_2, . . . , v_n \}$ and an edge set $E(G) = \{ e_1, e_2, . . ., e_m\}$, where each edge consists of two (possibly equal) vertices called its endpoints. We write uv for an edge e = (u, v). If $uv\in E(G)$, then u and v are **adjacent**. The ends of an edge are said to be **incident** with the edge. A **loop** is an edge whose endpoints are equal. **Parallel edges** or **multiple edges** are edges that have the same pair of endpoints. A **simple graph** is a graph having no loops or multiple edges. A graph is **finite** if its vertex set and edge set are finite. We adopt the convention that every graph mentioned in this paper is finite, unless explicity constructed otherwise. The **degree** of a vertex v of a graph G is the number of edges of G which are incident with v . A graph is said to be **regular** (**k-regular**) if all its vertices have the same degree (k). A three-regular graph is also called a **cubic graph**. A simple graph in which each pair of distinct vertices is joined by an edge is called a **complete graph**. A complete graph on n vertices is denoted by K_n. A sequence of distinct edges of the form$v_0v_1, v_1v_2, . . . , v_{r-1}v_r$ is called a **path of length r** from v_0 to v_r, denoted by (v_0, v_r)-path. The **distance** between two vertices u and v in a graph G is the length of the shortest path from u to v. The **diameter** of a graph is the maximum distance between two distinct vertices.

*. Dr., Lecturer, Department of Mathematics, Shwebo University

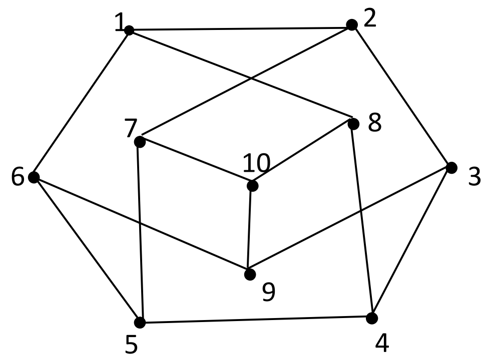

Figure 1.1: A graph G with diameter 2

A **subgraph** of a graph G = (V(G), E(G)) is a graph Y = (V(Y), E(Y)) with $V(Y) \subseteq V(G)$ and $E(Y) \subseteq E(G)$.Two vertices u and v of G are said to be **connected** if there is a (u, v)-path in G . A **connected graph** is a graph such that any two vertices are connected by a path, otherwise it is **disconnected**.

A **directed graph** (or **digraph**) $\vec{G} = (V(\vec{G}), E(\vec{G}))$ consists of a finite nonempty set $V(\vec{G})$, called **the set of vertices**, and set $E(\vec{G})$ of ordered pairs of (not necessarily distinct) vertices, called **the set of (directed) edges** or **arcs.** If $e=(u,v)$ or uv is a directed edge of $\vec{G}$, we say that e **joins** u to v, that u and v are **endpoints** of e(more specifically that u is the **tail** of e and v is the head of e). A digraph $\vec{G}$ is called **symmetric** if, whenever (u,v) is an arc of $\vec{G}$, then (v,u) is also. A digraph $\vec{G}$ is called **complete** if for every two distinct vertices u and v of $\vec{G}$, at least one of the arcs (u, v) and (v, u) is present in $\vec{G}$. A **complete symmetric digraph** of order n has both arcs (u,v) and (v,u) for every two distinct vertices u and v, denoted by $\vec{K}_n$.

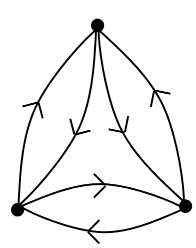

Figure 1.2: A complete symmetric digraph $\vec{K}_3$

2. Vertex-Transitive Graph

Before defining a vertex-transitive graph, we express the definition of an automorphism of a graph which plays a crucial role in determining the vertex-transitive graph.

An **isomorphism** from graph G to graph G' is a bijection $\phi : V(G) \to V(G')$ such that $uv \in E(G)$ if and only if $\phi(u)\,\phi(v) \in E(G')$. We say "G is isomorphic to G'", written $G \cong G'$, if there is an isomorphism from G to G'.

The graphs G and G' drawn below are isomorphic by an isomorphism that maps u, v, w, x, y, z to l, m, n, p, q, r respectively.

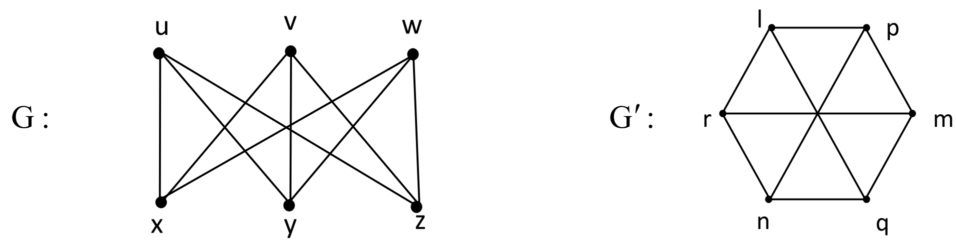

Figure 2.1: Isomorphic graphs G and G'

A **permutation** ϕ of V(G) is a function from V(G) into V(G) that is both one to one and onto.

An **automorphism** of a (simple) graph G is a permutation ϕ of V(G) which has the property that $uv \in E(G)$ if and only if $\phi(u)\,\phi(v) \in E(G)$, that is an isomorphism from G to G. The set of all automorphisms of a graph G forms a group under the operation of composition, which is called the **automorphism group**.

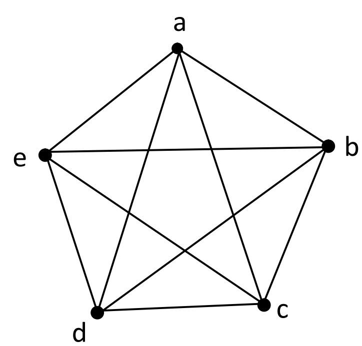

Figure 2.2: Complete graph K_5

In Figure 2.2, there is an automorphiosm ϕ of $V(K_5)$ such that $\phi(a) = b,\ \phi(b) = c,\ \phi(c) = d,\ \phi(d) = e,\ \phi(e) = a.$

For any two vertices u and v of G, there is an automorphism ϕ of G such that $\phi(u) = v,$ we say that G is **vertex-transitive.**

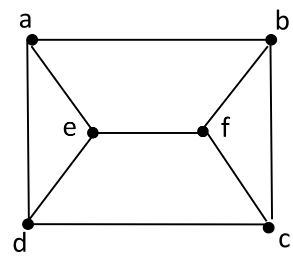

Figure 2.3: Vertex-transitive graph G

Now we interested in the structure of vertex-transitive graphs, in particular, Cayley graphs. First, we have to introduce some definitions of group theory.

3.Basic Definitions of Group Theory

A nonempty set of elements X is said to form a **group** if in X there is defined a binary operation, called the product and denoted by·, such that

(i) a, b $\in$X implies that a ·b $\in$ X (closed).

(ii) a, b, c $\in$ X implies that a · (b ·c) = (a· b) ·c (associative law).

(iii) There exists an element e$\in$ X such that a · e = e · a = a for all a $\in$ X (the existence of an identity element in X).

(iv) For every a $\in$ X there exists an element $a^{-1}\in$ X such that $a \cdot a^{-1} = a^{-1} \cdot a = e$ (the existence of inverses in X).

We usually write a b instead of a ·b.

A **finite group** is a group which has a finite number of elements, otherwise we call it an **infinite group**.

The number of elements in X is called the **order** of X and it is denoted by O(X)or $|X|$.We define the **order of an element** x to be the least positive integer n such that $x^n = e$ and we denote it by O(x) or $|x|$.

A nonempty subset S of a group X is said to be a **subgroup** of X if, under the product in X, S itself forms a group.

If S is a subgroup of X, $a \in X$, then $aS = \{as | s \in S\}$. a S is called a **left coset** of S in X.

Let X be a group of permutation of a set A and $b \in A$, then the **stabilizer** of b (in X) is the subgroup $X_b = \{x \in X \,|\, x(b) = b\}$.

A group X is called a **cyclic group** if there exists an element $x \in X$, such that every element of X can be expressed as a power of x. In that case x is called **generator** of X.

Let $D_n = \{x^i\, y^j \,|\, i = 0, 1;\ j = 0, 1, ..., n-1;\ x^2 = e = y^n, x\,y = y^{-1}x\}$. Then D_n is a group, called the **dihedral group**, $(n \geq 3)$. $O(D_n) = 2n$. In fact, we can write D_n also as

$$D_n = \{y,\ y^2, ...,\ y^{n-1},\ y^n,\ x\,y,\ x\,y^2, ...,\ x\,y^{n-1},\ x \,|\, x^2 = e = y^n,\ x\,y = y^{-1}x\}.$$

Let X be a group. A subset $H \subseteq X$ is a **generating set** of X if every element of X is obtainable as the product (or sum) of elements of H.

For the group Z_n, a nonempty set of integers modulo n is a generating set if and only if its greatest common divisor (gcd) is 1. For instance, the set {4,7} generates Z_{24}, since gcd (4,7) = 1 but {6,9} does not generate Z_{24}, since gcd (6,9) = 3.

If A is a finite set {1,2, . . . ,n}, then the group of all permutations of A is the **symmetric group on n letters**, and is denoted by S_n. Note that S_n has n! elements.

4. Cayley Graphs

Now we shall express the definitions of Cayley graphs and the constructions of Cayley graphs with their given groups.

4.1 Definitions

(i) Let X be a group and H a subset of X not containing the identity e. Then the **Cayley digraph** $\vec{C}$ has vertex set $V(\vec{C}) = X$ and arc set $E(\vec{C}) = \{(g, gh) \mid h \in H, \ g \in X\}$. We write $\vec{C} = \vec{C}(X, H)$.

(ii) Let $H = X - e$. Then the resulting Cayley digraph will be denoted by $\vec{K} = \vec{K}\ (X, H)$ and called the **complete Cayley digraph**.

(iii) Let X be a group and H a subset of X not containing the identity e such that $h \in H$ implies $h^{-1} \in H$ (that is, $H = H^{-1}$, where $H^{-1} = \{h^{-1} \mid h \in H\}$). Then the graph with vertex set $V(C) = X$ and edge set $E(C) = \left\{ (g, gh) \mid h \in H, g \in X \right\}$ is called the **Cayley graph** C corresponding to X, H. We write $C = C(X, H)$. Equivalently, the Cayley graph $C = C(X, H)$ is the simple graph whose vertex set and edge set are defined as follows:

$$V(C) = X; \ \ E(C) = \left\{ (g, h) \mid g^{-1} h \in H, \text{ where } g \in X, \ h \in H \right\}.$$

(iv) When H is a set of generators for X the Cayley digraph and the Cayley graph will be referred to as the **basic Cayley digraph** and the **basic Cayley graph** respectively.

4.2 Examples

(i) Let X be the group Z_5, the set of integers modulo 5.

Let H be generating set {1}.

We can construct the Cayley digraph $\vec{C} = \vec{C}(Z_5, H)$ which has the vertex set $V(\vec{C}) = Z_5 = \{0, 1, 2, 3, 4\}$; and the arc set $E(\vec{C}) = \{(0, 1), (1, 2), (2, 3), (3, 4), (4, 5)\}$.

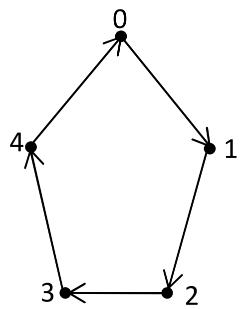

Figure 4.1: The Cayley digraph $\vec{C}(Z_5, \{1\})$

(ii) Let X be the cyclic group C4 generated by a and $H = \{a,\ a^2,\ a^3\}$.

We can construct the complete Cayley digraph $\vec{K} = \vec{K}(C_4,\ H)$ which has the vertex set $V(\vec{K}) = C_4 = \{1,\ a,\ a^2,\ a^3\}$; and the arc set

$$E(\vec{K}) = \{(1,a),(1,a^2),(1,a^3),(a,a^2),(a,a^3),(a,1),(a^2,a^3),(a^2,1),(a^2,a),(a^3,1),(a^3,a),(a^3,a^2)\}.$$

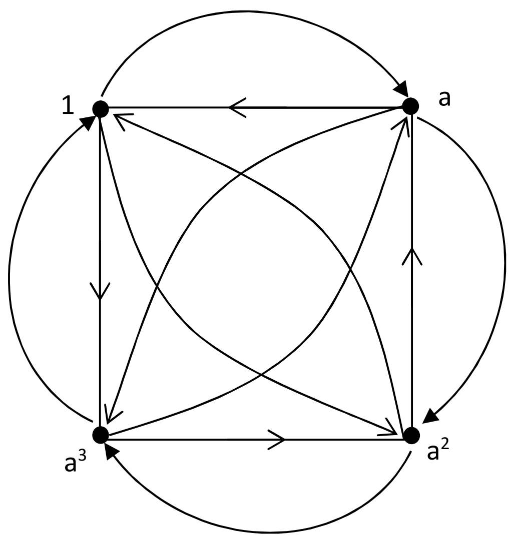

Figure4.2: The complete Cayley digraph $\vec{K}\ (C_4, \{a,\ a^2,\ a^3\})$

(iii) Let X be the symmetric group $S_3 = \{1,\ (12),\ (13),\ (23),\ (123),\ (132)\}$.

Let $H = \{(12),\ (13),\ (23)\}$.

Then $H^{-1} = H$. We can construct the Cayley graph $C = C(S_3, H)$ which has

$$V(C) = S_3;\ \ E(C) = \{(1,(12)), (1,(13)), (1,(23)), ((12), (132)), ((12), (123)), ((13), (132)), ((13), (123)), ((23), (132)), ((23), (123))\}.$$

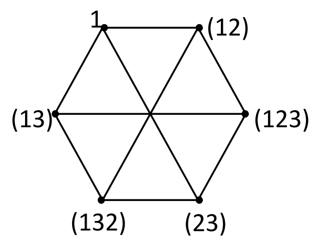

Figure4.3: The Cayley graph $C(S_3, \{(12), (13), (23)\})$.

4.3Theorem

The Cayley graph $C(X, H)$ is well-defined and is connected if and only if H is a set of generators for X.

Proof. See [7].

4.4 Examples

(i) Let X be the dihedral group $D_4 = \{1, r, r^2, r^3, s, rs, r^2s, r^3s\}$, where $r^4 = s^2 = 1$, $sr = r^{-1}s$ and $H = \{ r, s \}$.

Then H is a generating set for D_4. We can construct the Cayley digraph $\vec{C} = \vec{C}(D_4, H)$ has the vertex set $V(\vec{C}) = D_4$ and the arc set $E(\vec{C}) = \{ (g, gh) \mid g \in D_4,\ h \in H \}$.

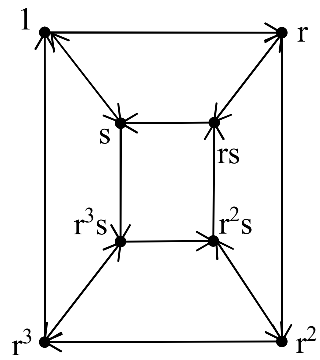

Figure 4.4: The connected Cayley digraph $\vec{C}(D_4, \{r, s\})$

(ii) Let X be $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and let $H = \{2, 6\}$.

Since $H = H^{-1}$ and H is not a generating set, we can construct the disconnected Cayley graph C has vertex set $V(C) = Z_8$ and edge set $E(C) = \left\{ (g, gh) \,\middle|\, g \in Z_8,\ h \in H \right\}$.

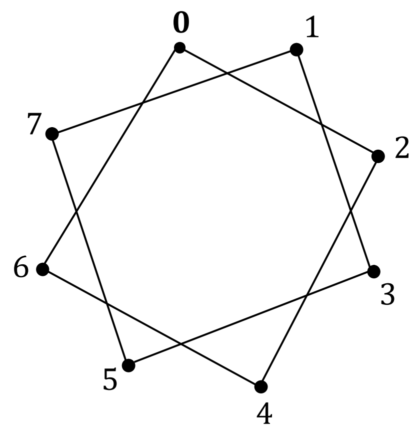

Figure 4.5: The disconnected Cayley graph $C(Z_8, H)$

In the above examples, we see that if the subset H is a generating set for the given group then the Cayley digraph is connected and if H is not a generating set then the Cayley graph is disconnected.

5. Relations between Cayley Graph and Vertex-Transitive Graph

In this section, we interested in a relation between Cayley graph and vertex-transitive graph.

5.1Theorem

Every Cayley graph $C(X, H)$ is vertex-transitive.

Proof.

For each g in X we define a permutation ϕ_g of $V(C) = X$ by the rule $\phi_g(h) = gh,\ h \in X$.

This permutation ϕ_g is an automorphism of C , for

$$(h,k) \in E(C) \Rightarrow h^{-1}k \in H$$

$$\Rightarrow (gh)^{-1}(gk) \in H$$

$$\Rightarrow (\phi_g(h), \phi_g(k)) \in E(C)$$

Now for any h, k $\in$ X, $\phi_{kh^{-1}}(h) = (kh^{-1})h = k$.

Hence Cayley graph $C(X, H)$ is vertex-transitive.

5.2 Petersen graph

The **Petersen graph** P(5,2) is a cubic graph having a vertex set $V = \{u_0,...,u_4, v_0,...,v_4\}$ and an edge set $E = \{(u_i,u_{i+1}),(u_i,v_i),(v_i,v_{i+2}) \mid i = 0,...,4\}$ where all the subscripts are taken modulo 5. The **generalized Petersen graph** $P(n, k)(n \geq 5,\ 0 < k < n)$ is the cubic graph having a vertex set $\{u_0,...,u_{n-1}, v_0,...,v_{n-1}\}$ and an edge set $\{(u_i,u_{i+1}),(u_i,v_i),(v_i,v_{i+2}) \mid i = 0,...,n-1\}$ where all the subscripts are taken modulo n.

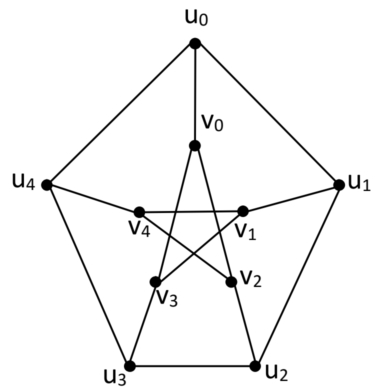

Figure5.1: The Petersen graph P(5, 2).

The following is an example of a vertex-transitive graph which is not a Cayley graph.

5.3 Example

The Petersen graph is vertex-transitive but it is not a Cayley graph.

Indeed, we can see the diameter of the Petersen graph is 2 and the diameter of a Cayley graph $C = C(X, H)$ is the smallest positive integer n such that $X = H \cup H^2 \cup ... \cup H^n$ where $H^2 = \{hk \mid h, k \in H\}$ and $H^i = H^{i-1}H$ for $i \geq 3$.

We now show that all the Cayley graphs of order 10 having degree 3 are of diameter greater than 2 and so none of them is the Petersen graph.

There are two groups of order 10. The first one is the cyclic group Z_{10} and the second one is the dihedral group D_5. The group operation here are additions and we replace H^{-1} by – H.

Case 1.

$X = Z_{10} = \{0, 1, ..., 9\}$.

Since – H = H and $|H| = 3$, $5 \in H$ and H can only be one of the following four sets

$H_1 = \{1, 5, 9\}$, $H_2 = \{2, 5, 8\}$,

$H_3 = \{3, 5, 7\}$, $H_4 = \{4, 5, 6\}$.

Now $|H_i + H_i| = 5$ for each $i = 1, 2, 3, 4$.

Thus the diameter of C is greater than 2.

Case 2.

$X = D_5 = \{0, b, 2b, 3b, 4b, a, a+b, a+2b, a+3b, a+4b\}$ where 2a = 0, 5b = 0 and b + a = a + 4b.

In this case a, a + b, a + 2b, a + 3b and a + 4b are the only elements of order 2 in X.

Hence H can only be one of the following three types of sets

$H_1 = \{a + jb, b, 4b\}$, $j = 0, 1, 2, 3, 4$;

$H_2 = \{a + jb, 2b, 3b\}$, $j = 0, 1, 2, 3, 4$;

$H_3 = \{a + j_1 b, a + j_2 b, a + j_3 b\}$, $0 \le j_1 < j_2 < j_3 \le 4$.

Now $|H_i + H_i| = 5$ for each i = 1, 2, 3.

Thus the diameter of C is greater than 2 also.

Petersen graph is a vertex-transitive graph but it is not a Cayley graph.

From the above example, we see that every vertex-transitive graph is not a Cayley graph. But every vertex-transitive graph can be constructed almost like a Cayley graph. This result will be shown in Theorem 5.5. We shall apply the following theorem to prove Theorem 5.5.

5.4 Theorem

Let S be a subgroup of a finite group X and let H be a subset of X such that $H^{-1} = H$ and $H \cap S = \varnothing$. If G is the graph having vertex set $V(G) = X/S$

(the set of all left cosets of S in X) and edge set $E(G) = \{(xS, yS) \mid x^{-1}y \in SHS\}$, then G is vertex-transitive.

Proof.

We first show that the graph G is well-defined.

Suppose that $(xS, yS) \in E(G)$ and $x_1S = xS,\ y_1S = yS.$

Then $x_1 = xs,\ y_1 = yk$ for some $s, k \in S.$

$$\text{Now}\quad x^{-1}y \in SHS \Rightarrow (xs)^{-1}(yk) \in SHS$$
$$\Rightarrow x_1^{-1}y_1 \in SHS$$
$$\Rightarrow (x_1s,\ y_1s) \in E(G).$$

Hence the graph G is well-defined.

Next, for each $g \in X$ we defined a permutation ϕ_g of $V(G) = X/S$ by the rule such that $\phi_g(xS) = gxS,\quad xS \in X/S$.

This permutation ϕ_g is an automorphism of G, for

$$(xS,\ yS) \in E(G) \quad \Rightarrow x^{-1}y \in SHS$$
$$\Rightarrow (gx)^{-1}(gy) \in SHS$$
$$\Rightarrow (gxS,\ gyS) \in E(G)$$
$$\Rightarrow \left(\phi_g(xS),\ \phi_g(yS)\right) \in E(G).$$

Finally, for any $xS,\ yS \in X/S$, $\phi_{yx^{-1}}(xS) = yx^{-1}(xS) = yS.$

Hence the graph G is vertex-transitive.

The graph G constructed in above theorem is called the **group-coset graph** X/S generated by H and is denoted by $G(X/S, H)$.

5.5 Theorem

Let G be a vertex-transitive graph whose automorphism group is A. Let $H = A_b$ be stabilizer of $b \in V(G)$. Then G is isomorphic with the group-coset graph $G(A/H, S)$ where S is the set of all automorphism x of G such that $\big(b, x(b)\big) \in E(G)$.

Proof.

We can see that $S^{-1} = S$ and $S \cap H = \varnothing$.

We now show that $\phi : A/H \to G$ given by $\phi(xH) = x(b)$, where $xH \in A/H$, defines a map.

Suppose $xH = yH$.

Then $y = xh$ for some $y \in H$.

$\phi(yH) = y(b) = (xh)(b) = x\big(h(b)\big) = x(b) = \phi(xH)$.

We next show that ϕ is a graph isomorphism.

Suppose $\phi(xH) = \phi(yH)$.

Then $x(b) = y(b)$

$$y^{-1}x(b) = b$$

$$y^{-1}x \in H$$

$$x \in yH$$

$$yH = xH.$$

So ϕ is one to one.

Let c be a vertex of G.

Since G is vertex-transitive, there exists z in A such that $z(b) = c$

Thus $\phi(zH) = z(b) = c$.

So ϕ is onto.

Next $(xH,\ yH) \in E\left(G(A/H,S)\right) \Leftrightarrow x^{-1}y \in HSH$

$$\Leftrightarrow x^{-1}y = hzk \text{ for some } h,k \in H,\ z \in S$$

$$\Leftrightarrow h^{-1}x^{-1}yk^{-1} = z$$

$$\Leftrightarrow \left(b,\ h^{-1}x^{-1}yk^{-1}(b)\right) \in E(G)$$

$$\Leftrightarrow \left(b,\ x^{-1}y(b)\right) \in E(G)$$

$$\Leftrightarrow \left(x(b),\ y(b)\right) \in E(G)$$

$$\Leftrightarrow \left(\phi(xH),\ \phi(yH)\right) \in E(G).$$

Thus G is isomorphic with the group-coset graph $G(A/H,\ S)$.

References

1. Alspach, B., "*CS E6204 Lecture 6 Cayley Graphs*", University of Regina, Canada, pp. 2-15,(2010).
2. Biggs, N., "Algebraic Graph Theory", Cambridge University Press, Cambridge, (1993).
3. Bondy, J.A. and Murty, U.S.R., "Graph Theory with Applications", The Macmillan Press Ltd, London, (1976).
4. Chen, T. Y., "*Connectivity in Vertex-Transitive Graphs*", M.Sc Thesis Submitted to Simon Fraser University, Canada, pp.27-38, (1994).
5. Herstein, I.N., "Topics in Algebra", John Wiley and Sons, Inc., New York, (1975).
6. Khanna, V. K. and Bhambri, S.K., "A Course in Algebra", Vikas Publishing House Pvt.,Ltd., New Delhi, (1998).
7. Parthasarathy, K.R., "Basic Graph Theory", Tata McGraw-Hill Publishing Company Limited, New Delhi, (1994).
8. Yap, H.P., "Some Topics in Graph Theory", Cambridge University Press, Cambridge, 1986.