

RESONANCE STATES OF α -n SYSTEM BY APPLYING COMPLEX SCALING METHOD

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Abstract

The purpose of this research is to study the resonance energy of ${}^5_2\text{He}$ (α , n) system by solving the two body Schrödinger equation with complex scaling method. The Gaussian basis wave function is used to solve the two body Schrödinger equation for ($J^\pi = 3/2^-$) state and ($J^\pi = 1/2^-$) state in spin-orbit coupling. The calculated resonance energies and level widths for ($J^\pi = 3/2^-$) state and ($J^\pi = 1/2^-$) state are (0.747, 0.596) MeV and (2.147, 5.533) MeV respectively. Comparison is made with the experimental data and good agreement is found.

Key words: resonance energy, complex scaling method, spin-orbit coupling

Introduction

The nuclear structure is roughly homogeneous distribution of neutrons and protons. Binding energy curve is the experimental evidence about the nuclear structure. The shape of this curve is the reflection of structure of nucleus. Hence every theory which evolved from 1930s till this time was trying to reproduce the binding energy curve. It started from the Liquid Drop Model (LDM), Alpha Clustering Model and now it reaches at the shell model, which is the most successful model in describing the structure of nucleus so far. However, the clustering phenomena is important to determine the structure of light nuclei.

LDM could fit smoothly with the binding energy curve and explain the fission mechanism successfully, but could not explain why there is a large binding energy for even nuclei such as ${}^4\text{He}$, ${}^{12}\text{C}$, etc. The success of the shell model is that it could explain the reason behind the large binding energy for even nuclei. According to the simplest shell model, every nuclei is spherical in

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structure. But many experimental results revealed that 90% of the nuclei are not spherical. Therefore, the extension of the shell model for the deformed nuclei shape is given first by S.G. Nilsson in 1955. Some experimental studies at CERN (Gaffney, 2013) show that the structure of Rn(Radon) is not spherical but pear shaped and they expect to see more such nuclei near Th(Thorium). These results show that there is a need for a new theory to well explain the structure of nuclei. There is the interaction between nucleons (as it is in reality) in the model approach, it can expect the formation of clusters. Clustering is a natural energy minimization mechanism. There is a large scale, it can see Galaxy clusters, star clusters, planets, etc. There are the small scales, it can also see clustering, quarks clustered to form molecules.

The idea of alpha clustering has a history back to 1930s. By observing alpha decay from nucleus, people speculated that nuclei are made up of alpha particles. ^8Be has one bond between two alpha particles and they dumbbell-like shape, ^{12}C has three alpha bonds and triangular shape, and ^{16}O has six alpha bonds and tetrahedral shape.

Resonances

The first resonance in particle physics was discovered by H. Anderson, E. Fermi, E. A. Long, and D. E. Nagle, working at the Chicago Cyclotron in 1952. Resonance states are formed when quantum particles collide at certain (resonant) energies. Before moving apart, they stay together for a while. During the resonance lifetime, the particles move around each other and “forget” the direction from which they came. Therefore, when the resonance eventually decays, the particles “choose” the direction to move away at random.

A resonance can be viewed and approached from two different angles, as a delay connected with an enhanced phased shift in a scattering process or as a long-lived but decaying state of a compound system. The main observable characteristics of a resonance are position and the width. The real and imaginary parts of the energy give the position and width of the resonance, respectively.

A resonance energy is $E_{res} = E_r - i\frac{\Gamma}{2}$ has a negative imaginary part, which is called resonance width. The use of a complex energy allows a classification of the energy levels of a quantum system.

Spin-Parity States of ${}^5_2\text{He}$ Resonance

In shell-model, shells are filled according to Pauli principle. This independence particle model further assumes a strong spin-orbit coupling so that each individual nucleon has a total angular momentum $j = \ell \pm \frac{1}{2}$ except in case $\ell = 0$ when j has only one value $\ell + \frac{1}{2} = \frac{1}{2}$. Thus each energy levels splits up into two sub-levels with $j \uparrow = \ell + \frac{1}{2}$ (s and ℓ parallel) and $j \downarrow = \ell - \frac{1}{2}$ (s and ℓ anti-parallel). The sub-level $j + \frac{1}{2}$ has a lower energy than the sub-level $j - \frac{1}{2}$. Each sub level of $N=1$ can have a maximum of $(2j+1)$ nucleons of the same kinds.

For odd mass nuclei, spin and parity is determined by orbital of the last unpaired nucleon. Parity is related to the orbital quantum number ℓ and is given by $P = (-1)^\ell$.

The relative orbital angular momentum of alpha and nucleon is 1. Spin and parity of lowest lying states are $(3/2^-)$ state and $(1/2^-)$ state and shown in Fig. (1).

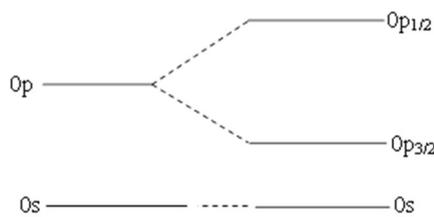


Figure 1. Energy levels of unpaired neutron in ${}^5_2\text{He}$ system

Complex Coordinate Rotation Method

The method of complex coordinate rotation is used to investigate the resonance states. Under this treatment a resonance is considered as an extension of the concept of a bound state which is solution to the Schrödinger equation with purely outgoing asymptotic belonging to the complex energy. The real and imaginary parts of the energy give the position and width of the resonance, respectively. Thus complex resonance energy is written as

$$E_{\text{res}} = E_R - i \frac{\Gamma}{2} \quad (1)$$

where E_R is energy level and Γ is the level width. In the asymptotic region, the resonance wave function is described by purely outgoing radial part as

$$\Psi(r) \xrightarrow{r \rightarrow \infty} e^{iKr} \quad (2)$$

where the wave vector $K = K_R - iK_I$ ($K_R > 0$ and $K_I > 0$) and satisfies

$$E = \frac{\hbar^2 K^2}{2\mu}.$$

Thus

$$\begin{aligned} \Psi(r) \xrightarrow{r \rightarrow \infty} e^{i(K_R - K_I)r} &= e^{iK_R r} e^{-K_I r} \\ &= e^{K_I r} \{ \cos(K_R r) + i \sin(K_R r) \} \end{aligned} \quad (3)$$

which shows that the asymptotic radial part is oscillating between exponentially growing amplitudes of $e^{K_I r}$.

The asymptotic divergence of the resonant wave function has caused difficulties in the resonance calculations. This resonance wave function cannot be solved by using the bound state type wave function.

According to complex rotation method, the following transformation

$$r \rightarrow r e^{i\theta} \quad (4)$$

where θ is a rotation angle. The complex scaling operator $\hat{U}(\theta)$ acting on a single particle wave function is defined as

$$\Psi(r) \rightarrow \Psi(re^{i\theta}) = \Psi_{\theta}(r) = \hat{U}(\theta) \Psi(r) \tag{5}$$

Let us see the transformed Schrödinger equation under complex rotation,

$$H(r) \Psi(r) = E \Psi(r) \tag{6}$$

$$H(re^{i\theta}) \Psi(re^{i\theta}) = E \Psi(re^{i\theta}) \tag{7}$$

$$\begin{aligned} H(re^{i\theta}) \hat{U}(\theta) \Psi(r) &= E \hat{U}(\theta) \Psi(r) \\ \hat{U}^{-1}(\theta) H(re^{i\theta}) \hat{U}(\theta) \Psi(r) &= E \hat{U}^{-1}(\theta) \hat{U}(\theta) \Psi(r) \\ \hat{U}^{-1}(\theta) H(re^{i\theta}) \hat{U}(\theta) \Psi(r) &= E \Psi(r) \end{aligned} \tag{8}$$

By comparing Eq.(6) and (8)

$$\begin{aligned} H(r) &= \hat{U}^{-1}(\theta) H(re^{i\theta}) \hat{U}(\theta) \\ \text{(or)} \quad \hat{U}^{-1}(\theta) H(r) \hat{U}(\theta) &= H(re^{i\theta}) = H_{\theta}(r) \end{aligned} \tag{9}$$

Under this transformation, the energy eigenvalue remains unchanged

$$H_{\theta}(r) \Psi_{\theta}(r) = E \Psi_{\theta}(r)$$

Then the asymptotic resonance wave function is transformed as

$$\begin{aligned} \Psi_{\theta}(r) &\xrightarrow{r \rightarrow \infty} e^{i(K_R - iK_I)r} e^{i\theta} \\ &\xrightarrow{r \rightarrow \infty} e^{-(K_R \sin \theta - K_I \cos \theta)r} e^{i(K_R \cos \theta + K_I \sin \theta)r} \end{aligned} \tag{10}$$

and oscillating with amplitude $e^{-(K_R \sin \theta - K_I \cos \theta)r}$. From the above equation, if $K_R \sin \theta > K_I \cos \theta$ (or) $\tan \theta > \frac{K_I}{K_R}$, the resonance wave functions becomes convergent at the asymptotic region.

Thus the resonance state can be solved with bound state type wave functions. According to the relation of the complex resonance energy and wave vector,

$$E_R - iE_I = \frac{\hbar^2}{2\mu} (K_R - iK_I)^2 \quad (11)$$

If $E_R \gg E_I$ ($K_R \gg K_I$), the boundary condition for resonance state can be expressed as

$$\tan \theta > \frac{K_I}{K_R} \sim \frac{E_I}{2E_R}$$

The rotation parameter θ has an upper limit, $\theta_c = \pi/4$. The Gaussian basis wave functions are transformed under complex rotation.

By introducing r' as $r' = r e^{i\theta}$ (12)

$$r = r' e^{-i\theta} \quad (13)$$

Eq. (7) is written as $H(r e^{i\theta}) \Psi(r e^{i\theta}) = E \Psi(r e^{i\theta})$

$$H(r') \Psi(r') = E \Psi(r') \quad (14)$$

The transformed Gaussian basis wave function is

$$\Psi_\theta(r) = \sum_j c_j(\theta) e^{-\left(\frac{r}{b_j}\right)^2} \quad (15)$$

(or) $\Psi(r') = \sum_j c_j(\theta) e^{-\left(\frac{r'}{b_j e^{i\theta}}\right)^2}$ (16)

Eq. (14) becomes

$$H(r') \sum_j c_j(\theta) e^{-\left(\frac{r'}{b_j e^{i\theta}}\right)^2} = E \sum_j c_j(\theta) e^{-\left(\frac{r'}{b_j e^{i\theta}}\right)^2} \quad (17)$$

$$H(r) \sum_j c_j(\theta) e^{-\left(\frac{r}{b_j e^{i\theta}}\right)^2} = E \sum_j c_j(\theta) e^{-\left(\frac{r}{b_j e^{i\theta}}\right)^2} \quad (18)$$

We have to solve the Schrödinger equation which is the same as bound state system except the range parameter b_j becomes $b_j e^{i\theta}$.

The bound states are the same. The positive-energy spectrum of the original Hamiltonian H is rotated down by an angle of 2θ into the complex-energy plane. The resonance states of the eigenvalues $E_{\text{res}} = E_R - i\frac{\Gamma}{2}$ satisfying the condition $E_{\text{res}} \ll 2\theta$, where E_R is the energy and Γ is width of the resonance, respectively. Where the bound states that lies on the negative energy axis remains unchanged, the resonance states located in the fourth quadrant, after the separation from the continuous spectrum, does not vary with θ , the continuum along the positive energy axis rotates clockwise by the angle 2θ . If we do not apply the complex scaling, the original Schrödinger equation gives the continuum spectra, including resonances on the positive energy axis.

Under complex rotation, the resonance for which $\tan\theta > \frac{k_I}{k_R}$ are separated from continuum, and the rotated continuum spectra starting from different threshold energies are separately obtained on different 2θ -lines as shown in Fig. (2).

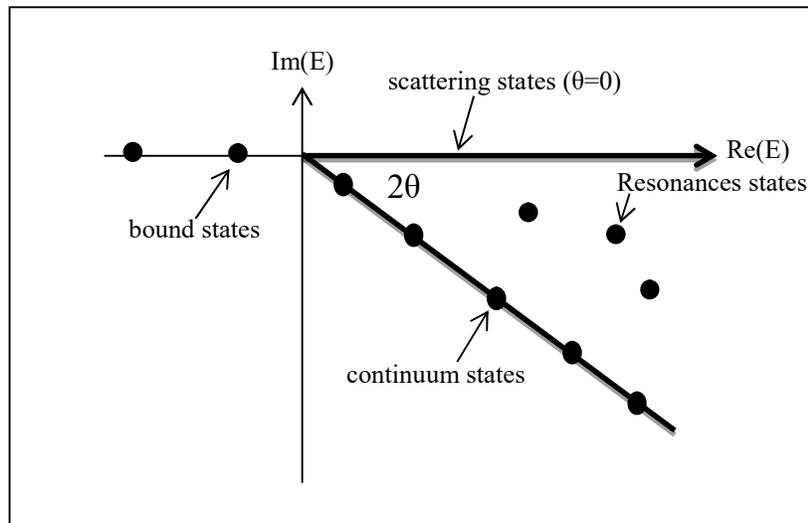


Figure 2. A schematic distribution of the eigenvalues of the bound states and resonance states.

Two-Body Calculation

We use the Gaussian basis wave function as the total wave function of our two-body system which has the following form;

$$u(r) = r^{\ell+1} \sum_j c_j e^{-\left(\frac{r}{b_j e^{i\theta}}\right)^2} \quad (19)$$

where c_j 's expansion coefficient and b_j 's are range parameters.

The Schrödinger equation is written as

$$H\Psi = E\Psi \quad (20)$$

The two body Hamiltonian is

$$H = -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{\hbar^2}{2\mu} \frac{\ell(\ell+1)}{r^2} + V(r)$$

where reduce mass , $\mu = \frac{m_1 m_2}{m_1 + m_2}$

And then we can write as;

$$\sum_{ij} [T_{ij} + V_{ij}^{\ell} + V_{ij}] c_j = E \sum_{ij} N_{ij} c_j \quad (21)$$

$$\sum_{j=1}^N H_{ij} c_j = E \sum_{j=1}^N N_{ij} c_j \quad (22)$$

Eq.(22) can be written as

$$H_{11} C_1 + H_{12} C_2 + \dots + H_{1N} C_N = E(N_{11} C_1 + N_{12} C_2 + \dots + N_{1N} C_N)$$

$$H_{21} C_1 + H_{22} C_2 + \dots + H_{2N} C_N = E(N_{21} C_1 + N_{22} C_2 + \dots + N_{2N} C_N)$$

.

$$H_{N1} C_1 + H_{N2} C_2 + \dots + H_{NN} C_N = E(N_{N1} C_1 + N_{N2} C_2 + \dots + N_{NN} C_N)$$

The above N equations can be written as a matrix form as follows;

$$\begin{bmatrix} H_{11} & H_{12}\dots & H_{1N} \\ H_{21} & H_{22}\dots & H_{2N} \\ \dots & \dots & \dots \\ H_{N1} & H_{N2}\dots & H_{NN} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \dots \\ C_N \end{bmatrix} = E \begin{bmatrix} N_{11} & N_{12}\dots & N_{1N} \\ N_{21} & N_{22}\dots & N_{2N} \\ \dots & \dots & \dots \\ N_{N1} & N_{N2}\dots & N_{NN} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \dots \\ C_N \end{bmatrix}$$

$$[H][C] = E [N][C]$$

$$[N]^{-1}[H][C] = E[C]$$

$$[A][C] = E[C]$$

$N_{ij}, T_{ij}, V^{\ell}_{ij}$ are analytically solved by using standard integral. We solved the two body Schrödinger equation to obtain the resonance energies. By diagonalization the Hamiltonian matrix elements, we obtained the complex energy eigenvalues.

Interaction between Alpha and neutron

We have employed the α -n potential which is introduced by Kanada *et al.*, (Kanada, 1979). The α -N potentials are described in the following parity dependent form with the central and spin-orbit terms. Spin-orbit coupling is predominant feature. The α -n potential is expressed in the following Gaussian form,

$$V_{\alpha-n}(r) = \sum_i^{i_{\max}} V_i e^{-\beta_i r^2} + \sum_i^{i'_{\max}} (-)^{\ell} V_i^p e^{-\beta_i^p r^2} + \left[\sum_i^{i''_{\max}} V_i^{\ell,s} e^{-\gamma_i r^2} + i \sum_i^{i'''_{\max}} (-)^{\ell} V_i^{\ell,s,p} e^{-\gamma_i^p r^2} \right] \vec{\ell} \cdot \vec{S}_N \quad (23)$$

where ℓ is the relative angular momentum between α and N, and \vec{S}_N is the spin of N. The spin orbit $\vec{\ell} \cdot \vec{s}$ coupling is calculated by using the following equation;

$$\vec{\ell} \cdot \vec{s} = \frac{j^2 - \ell^2 - s^2}{2} \quad (24)$$

$$\vec{\ell} \cdot \vec{s} = \frac{j(j+1) - \ell(\ell+1) - s(s+1)}{2} \quad (25)$$

The potential strengths and the range parameters are shown in Fig. (3), Fig. (4) and described in Table (1).

Table 1. The size parameters (fm^{-2}) and strength parameters (MeV) of α -n interaction

i	1	2	3
β_i	0.36	0.90	-
β_i^p	0.20	0.53	2.5
γ_i	0.396	0.52	2.2
γ_i^p	0.396	2.2	-
V_i	-96.3	77.0	-
V_i^p	34.0	-85.0	51.0
V_i^{ls}	-20.0	-16.8	20.0
$V_i^{ls,p}$	6.0	-6.0	-

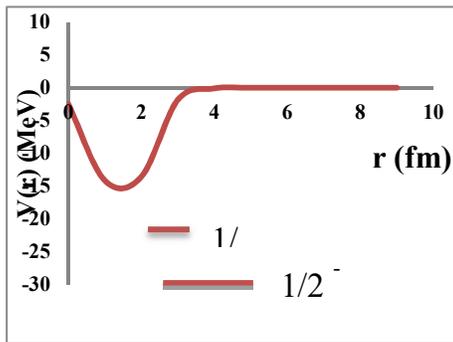


Fig. (3)

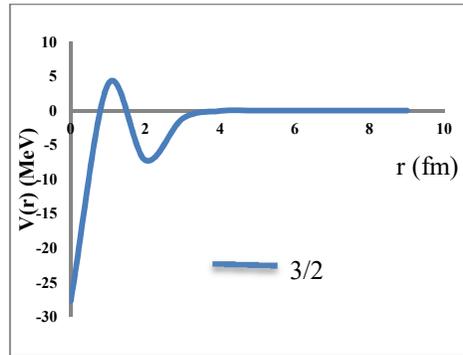


Fig. (4)

Figure 3. The potential for (α -n) system in ($J^\pi = 1/2^-$) state.

Figure 4. The potential for (α -n) system in ($J^\pi = 3/2^-$) state.

Results And Discussion

Resonance Energies of ${}^5\text{He}$ System

Our purpose is to investigate the resonance states of α -n system by using the complex scaling method. The (α -n) system has rather broad resonances but no bound state. The complex-scaled Hamiltonian matrix

elements are diagonalized and we obtained the complex energy eigenvalues. The number of the basis state is determined to converge the solutions.

The complex eigenvalues $E = E_r - i\Gamma/2$ the energy level E_r and level width Γ . From the energy eigenvalue data, we have found that the difference between resonance states and continuum states increases with increasing θ . In complex scaling method, the continuum states vary with the 2θ dependence. This distribution of continuum eigenvalues depends on the choice of the number of basic. In our calculation, the optimum set of parameters are $G_1=0.1\text{fm}$, $CG=1.15$, $N=40$. The resonance states for $(J^\pi = 3/2^-)$ can be seen clearly from the continuum states $\theta=20^\circ$ to 40° . The energy eigenvalue distributions of the ${}^5\text{He} (3/2^-)$ state with $\theta=20^\circ$, $\theta=30^\circ$ and $\theta=40^\circ$ are shown in Fig. (5). The energy eigenvalue distributions of the ${}^5\text{He} (1/2^-)$ state with $\theta=30^\circ$, $\theta=35^\circ$ and $\theta=40^\circ$ are shown in Fig. (6). From our calculation, we found that bound states and resonance states are discrete and obtained independently of θ . The calculated energies and level widths of ${}^5\text{He}(\alpha - n)$ for $(J^\pi = 3/2^-)$ and $(J^\pi = 1/2^-)$ are (0.747, 0.596)MeV and (2.147, 5.533)MeV respectively. Resonance energies for $(\alpha - n)$ system are compared with the experimental results (D.R.Tilley, 2002) in Table (2). Energy levels diagram for each state by comparing with the experimental values are shown in Fig. (7).

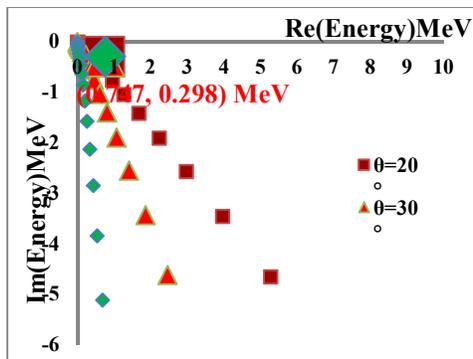


Figure 5

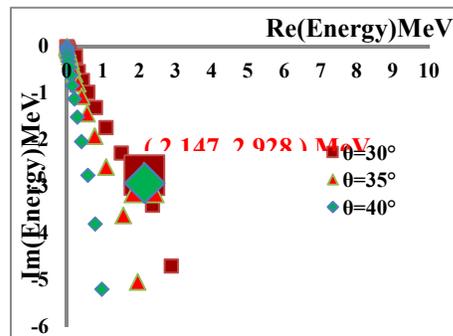


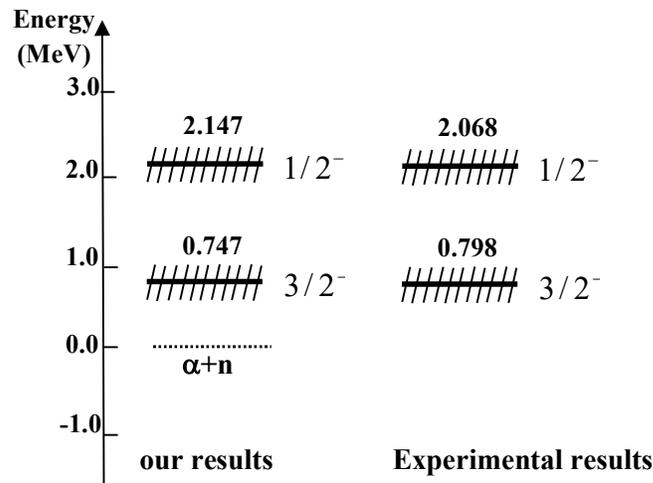
Figure 6

Figure 5. Resonance energy for the ${}^5\text{He} (3/2^-)$ with $\theta=20^\circ$, $\theta=30^\circ$, $\theta=40^\circ$

Figure 6. Resonance energy for the ${}^5\text{He} (1/2^-)$ with $\theta=30^\circ$, $\theta=35^\circ$, $\theta=40^\circ$

Table 2. The energy eigenvalues of ${}^5\text{He}$ system.

States	Energy Level, E (MeV)		Level Width, Γ (MeV)	
	Our Results	Exp: Results (D.R. Tilley, 2002)	Our Results	Exp: Results (D.R. Tilley, 2002)
${}^5\text{He}(3/2^-)$	0.747	0.798 \pm 0.008	0.596	0.648 \pm 0.006
${}^5\text{He}(1/2^-)$	2.147	2.068 \pm 0.021	5.533	5.570 \pm 0.056

**Figure 7.** Energy level of ${}^5\text{He}$ system in comparison with experimental values

Conclusion

Studies of resonances are indispensable for understanding the unique properties of drip-line nuclei. We have calculated the $\alpha+N$ two body calculation for ${}^5\text{He}$ resonance states by using complex rotation method. From our calculation, we found that the positions of resonance states in the complex energy plane remains almost unchanged with the variation of rotation angle θ . Our calculated results for the energies and level widths of ${}^5\text{He}$ system are consistent with the experimental results.

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